PETROLEUM REFINERY OPERATIONS SCHEDULING

CHAPTER V - A GASOLINE BLENDING PROBLEM

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1. Basic economic assumptions

The preceding chapter has dealt with one type of intra-refinery economic balance problem -- the operation of a typical kind of conversion unit. The present one concentrates upon another such intra-refinery study -- the selection of an optimal gasoline blending schedule. Not until the following chapter will any attempt be made to discuss the operations of an integrated refinery.

There were two primary reasons for singling out the blending problems of the Union Oil Company of California: First, this operation is itself a complex large-scale one -- grossing an amount of the order of $100 millions annually. And second, the data available for this analysis were of tolerable precision. It was agreed that there is little point in making an elaborate economic study when the basic numbers   are shaky.

Together with members of the Manufacturing Economics Division of Union Oil, the problem was formulated in the following terms (see the schematic diagram, Figure 10): First, the quantities and qualities of 19 raw gasoline stocks were assumed given. Together with tetraethyl lead (a purchased item), these serve as the inputs to the gasoline blending operation. The input materials are combined to form three basic types of products: 7500, premium grade gasoline; 76, regular grade gasoline; and fuel oil cutter stock. As a first approximation, it was assumed that each of these products could be sold at refinery realizations that were independent of the total quantity marketed. The scheduling problem consists of assigning each raw gasoline stock among

the three potential products, and also of setting the tetraethyl lead levels
for the two motor fuels. This assignment has to take into account not only
the absolute quantities of materials that are available, but also certain
product specifications on the two finished gasolines.

Each of these assumptions must be reviewed with care. Consider, for
example, the matter of blending stocks. With the exception of just two of
the 1 inputs, it is assumed here that the quantities of raw gasolines avail-
able and that the quality characteristics of these materials are independent
of decisions taken in the blending department. In fact, from the viewpoint
of a refinery superintendent, the quantities and characteristics of all the
streams are variables subject to control. By altering the choice of crudes,
the reactor temperatures, the recycle ratios, and the assignment of inter-
mediate oil streams, the central management is able to influence the size
and composition of the gasoline blending materials. This type of interlock
can only be studied in a larger model that cuts across departmental lines.

A second point   is the assumption that
additional quantities of finished refinery products may be marketed at the
prevailing realizations. In the case of a major refiner, this is open to
serious doubt. It is not at all obvious that a large company can put additional
quantities on the market without disturbing the current price structure.[3]

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[2] During the course of short time intervals, the quantities and qualities of
the available blending stocks may change radically. An unforeseen shutdown
of a unit can take place. A new unit may be brought on stream. A new source
of crude oil may be introduced, etc. For these reasons, in order to keep
blending analyses up to date, it is the current practice within the company
to recalculate schedules at least once a month. The optimum allocations of
raw streams among the various uses could be significantly affected if, say,
a fire occurred in a large catalytic cracker.

[3] See J. S. Bain, The Economics of the Pacific Coast Petroleum Industry,
University of California Press, Part I, 1944; Part II, 1945; and Part III, 1947.

At the same time, no one -- least of all an economist -- is in a position to make a reasonable quantitative estimate of the relationship between a company's price structure and the demand for its products.

For purposes of computation, an explicit marketing limitation could have been placed upon each finished product. In order to provide a benchmark, however, it was decided to study one basic case, and then one in which a sales restriction was introduced. The comparison of these two provides an indication of the extent to which profits could be improved by overcoming the marketing barrier. The comparison also indicates the magnitude of the gap between the refinery realization and the incremental cost in the second instance.

Corresponding to the assumption about the realization upon refinery products, it was agreed that one of the inputs -- tetraethyl lead -- would be considered as available without any quantity restrictions at the Ethyl Corporation's prevailing tank-car delivered price. During wartime, TEL has been rationed among refiners, but at least during 1953, this type of restriction did not, hold true. If lead were to become short again, the explicit availability limitation would have to be considered as an element in the problem. As shown in the preceding chapter, it is possible for the refiner to substitute reformed and cracked gasolines over a wide range in place of tetraethyl lead in the finished products. Although lead is employed in virtually every gallon of gasoline today, its use in motor fuel could be eliminated -- but only at a distinct cost in dollars and cents to the refiner.

Having decided upon the two major assumptions -- fixed gasoline stock availabilities and fixed refinery product realizations -- the next step was to determine which of the numerous quality characteristics of the finished products

would be studied explicitly. The Manufacturing Economics Division pointed
out that there were many specifications that finished motor fuel must meet --
three types of octane numbers; 10%, 30%, 60%, 90%, and end points of the
boiling range; sulfur and gum content; vapor pressure, etc. Any of these
could conceivably limit the output of either premium or regular grade gasoline.
For purposes of this study,   the ones that seemed to be worth the most
close attention were: Research octane number, the percent sulfur content, and
a volatility index. In addition, the end point problem was handled auto-
matically by limiting the end points of the individual raw gasolines. During
the course of the computing runs, it turned out that sulfur was not a limiting
specification,   but that the octane number and volatility index specifications
did come into play.

In order to protect proprietary information, the exact house-brand speci-
fications cannot be revealed here. It can, however, be said that the 7600
premium gasoline has a higher minimum octane number and volatility index --
and a lower maximum permissible sulfur content and end point than the regular
76 gasoline. On both products, the maximum permissible tetraethyl fluid
concentration is 3.0 milliliters per gallon.

Although the blending section must operate within whatever company speci-
fications are currently effective. the reader should not suppose that these
limits remain fixed over time. At any moment, "octane wars" between refiners
can break out. The effect upon motorists -- and upon refinery profits -- is
of much the same nature as an actual price war. For this reason. a separate
computation was set up in order to determine the effects of a 2.5 octane
number increase in the specifications of the premium gasoline.

Before proceeding to the technology of the problem, two more of the

economic assumptions should be brought out -- one having to do with the

are manufactured,

cutter stocks and the other with transportation costs. The raw gasolines ∧

and the blending operations are performed at two   distinct locations --

at Wilmington in the Los Angeles area and at Oleum on San Francisco Bay.

Raw stocks are continually lifted by tanker for interchange between the two

refineries. Union Oil estimates its tanker costs between the two points

to be nominal - only a few percent of the refinery realization on premium

gasoline. In the analysis of the company's gasoline blending problems,

it is evidently a desirable thing to take account of transportation costs

and of differences in product realizations at the two refineries. At the

same time, it was easy to see that by bringing geography into the problem,

the volume of computations would have increased by more than 100%. The

storage capacity of the then available computing machine -- the International

Business Machine Card Programmed Calculator -- would have been completely

inadequate to take care of the enlarged model.[4] As a first effort, therefore,

it was agreed that the geography should be left out of account.

considered to be

The raw gasoline stocks were∧available in one company-wide pool,

were

and   the finished products∧   valued at the company's overall

realizations. (Roughly speaking, these revenue figures are of the same order

of magnitude as the current spot market refinery tank-car quotations.)

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4. This type of limitation would not have existed if one of the higher-
capacity electronic computers had been available, e.g., the IBM 701 or
the Univac.

there is

Finally, the problem of cutter stock realizations. As indicated by Figure 10 , eight of the heavy gasolines may be employed, not only for motor fuel components, but also for blending with heavy residuum into No. 6 fuel oil. The figure also shows that this heavy residuum may be converted into No. 6 oil by the alternative route of blending with No. 2 furnace oil.[5] Under the assumption that some heavy residuum is always going to be cut down to the viscosity specifications by blending with No. 2 oil, it is possible to establish a value for the heavy residuum, and then to go on to determine a cutter value for any particular heavy gasoline.

To give a purely hypothetical illustration of the procedure, assume the following product values:

No. 2 furnace oil $4.00/B.

No. 6 fuel oil $2.00/B

$ of No. 2 oil in No. 6 blend required for cutting back heavy residuum to No. 6 oil specifications 25%

$ of heavy gasoline in No. 6 blend required for cutting back heavy residuum to No. 6 oil specifications 20%

If No. 2 oil is used for blending, the value of the heavy residuum, p_r, may be established through the following equation:

Value of No. 6 oil = Value of No. 2 oil component + Value of residuum component

$2.00 = .25($4.00) + (1 - .25)(p_r)

p_r = $1.33/B

[5] In general, the cutter will not be a finished No. 2 oil, but rather some gas oil component that could be used for No. 2. When there are several alternative gas oils that may be employed for cutter, the problem takes on additional complexity. See Chapter VI below, p. VI - 26
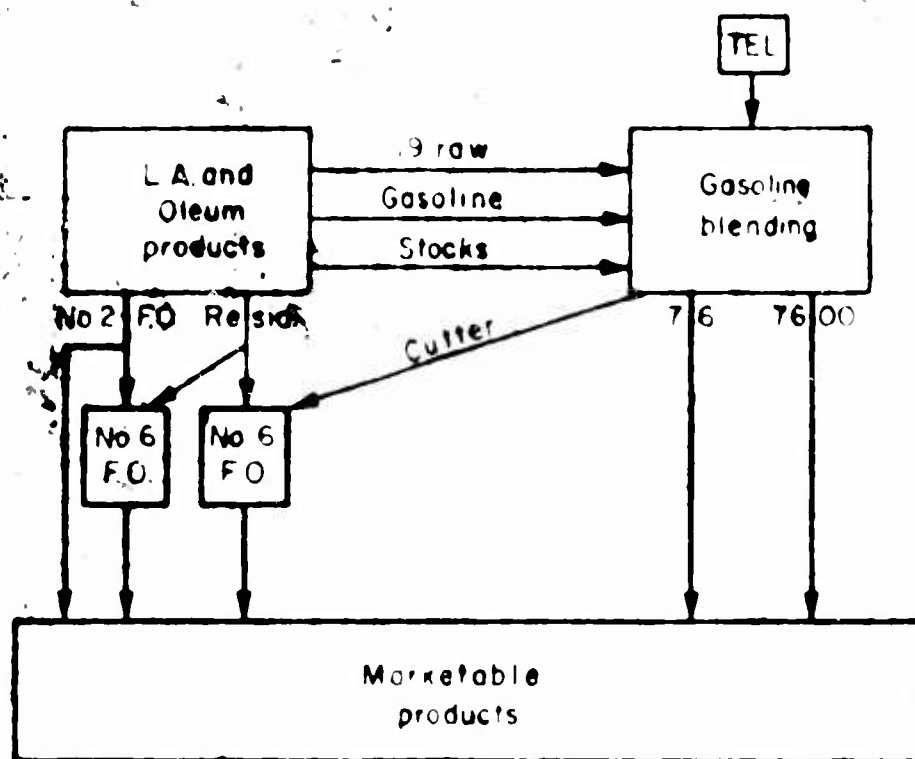
Fig. 10—Union Oil Co. gasoline blending problem

Next, this value for p_r is substituted in a costing equation for the heavy gasoline cutter stock. Out of this relation, there is derived P_g, the cutter value of the particular gasoline.

Value of No. 6 oil	=	Value of gasoline component	+	Value of residuum component
$2.00	=	.20(p_g)	+	(1 - .20)($1.333)
p_g	=	$4.668/B		

Values for each of the eight possible gasoline materials were calculated in this manner. Certain treatment cost savings also had to be credited to the cutter operation. The final calculated net cutter values range from 68.2% to 87.4% of the refinery realization on premium gasoline. In effect, this is an opportunity cost calculation of the same nature as the "gasoline replacement" technique described above, pp. II-5-7. Like the gasoline replacement formula, this method applies to the comparison of just two alternative refining operations. (In the one case, the alternatives are cracking versus fuel oil, and in the other, gasoline versus No. 2 fuel oil cutter materials.) In the event that the company's residuum fuel were capable of meeting the oil specifications without any additional cutter, the heavy gasoline could not be given the premium value as cutter, but would have to be assigned a lower credit.

2. The blending technology

Of the three relevant gasoline properties, two -- the volatility index and the percent sulfur content -- give rise to straightforward linear blending relationships. For example, in a 50-50 blend of two raw gasolines with sulfur contents of 0.10% and 0.30%, the resulting mixture will contain 0.20% sulfur. Similarly with the volatility index, a 50-50 blend will have an index that lies midway between those of its two component raw gasolines.

Unfortunately, the octane number of blends cannot be predicted in such a direct manner from the octane numbers of the components. Like most kinds of engineering work, the forecast of gasoline octane numbers is an rather art than an exact science. Refiners themselves necessarily make paper predictions of the octane number of proposed blends, but before marketing a product they will almost inevitably take the precaution of testing the mixture in an octane rating engine.

One phase of the problem has been investigated extensively by the refining industry -- the relationship between octane numbers and the TEL concentration of a particular gasoline. Just as in the naphtha reforming problem above, it has been observed that, as the lead concentration level is increased, the octane number increases at a decreasing rate. In order to predict this relationship, there is in widespread use today an ethyl blending nomogram put forward by Hebl, Rendel, and Garton in 1939.[6] There is, fortunately, a three-parameter analytic expression that gives a close approximation to the results predicted by this blending chart. Where t

6. L. E. Hebl, T. B. Rendel, and F. L. Garton, "Ethyl Fluid Blending Chart for Motor-Method Octane Numbers", Industrial and Engineering Chemistry, Vol. 31, July, 1939, pp. 862-865.

represents the octane number of the leaded gasoline, and $\underline{x}$ the TEL concentration level (in milliliters per gallon):

$$(V.1) \quad t = a + bx - \frac{c}{1+x}$$

For any unknown gasoline, the three positive parameters, a, b, and c, have to be determined. This may be accomplished readily by taking three observations of octane number for various lead levels, and solving the resulting simultaneous linear equations. Using equation (V.1) in this way, sets of constants were calculated for all 19 of the Union Oil Company's raw gasolines. Testing seven lead levels in each of these 19 cases, the equation has virtually always given a prediction that lies within $\pm$ 0.2 octane number of the one yielded by the blending chart.

No special significance should be attached to the particular form of equation (V.1). It is merely a device for enabling a computing machine to perform the same calculation as a refiner with his ethyl chart and his straight edge.

A second phase of the blending problem is more controversial than the TEL aspect. For a lead level of, say, 3.0 ml., and for a 50-50 mixture of two gasolines A and B, refiners frequently calculate the octane number of the blend to be the 50-50 weighted average of the octane number of gasolines A and B -- each with 3.0 ml. The weighted average appears satisfactory for many stocks -- especially when both components are of a paraffinic nature. There are, however, at least two papers publicly available -- one by Eastman[7] and the other by Bogen and Nichols[8] -- that call into question the straight-line average method.

[7] Du Bois Eastman, "Prediction of Octane Numbers and Lead Susceptibilities of Gasoline Blends", Industrial and Engineering Chemistry, Vol. 33, December, 1941, pp. 1555-1560.

[8] J. S. Bogen and R. M. Nichols, "Calculating the Performance of Motor Fuel Blends", Industrial and Engineering Chemistry, Vol. 41, November, 1949, pp. 2629-2635.

Both papers indicate that as the percentage of the high octane component in a binary mixture increases (both components initially at the same TEL concentration level), the octane level of the mixture may increase at a decreasing rate. In other words, the octane number of the blend tends to exceed the weighted average of the octane number of the two components.

Bogen and Nichols are primarily concerned with three-component mixtures, and they report only five observations at each of three TEL levels for strictly binary blends. The set of 15 points has been plotted, and is reproduced as Figure 11. For each TEL concentration level shown on this chart, straight-line interpolation has been used between adjacent observations. In all likelihood, these straight lines also underestimate the octane number that is attainable by blending. The existing data, unfortunately, do not provide a basis for curvilinear interpolation.

At first glance, it would appear to be a trivial matter whether octane numbers were calculated by drawing straight lines between two adjacent points or between two end points of a constant-TEL line. Nevertheless, a sample calculation presented in an earlier paper[9] indicates that the two methods can lead to TEL input requirements that differ by 20%.

It is evidently worthwhile for a refiner to incur considerable expense in testing gasoline blends, and in attempting to predict the occurrence of curvilinear octane relationships. Nevertheless, at the present stage, the Union Oil Company did not have such blending data available, and it was necessary to revert back to the more conventional straight-line interpolation procedure for calculating the octane numbers of blends.[10] The information (photo-offset

[9] A. S. Manne, "Concave Programming for Gasoline Blends", P-383
paper), The RAND Corporation, Santa Monica, California, April 23, 1953.

[10] In going back to the simple straight-line octane interpolation procedure, this paper follows the model proposed by A. Charnes, W. W. Cooper, and B. Mellon, "Blending Aviation Gasolines -- A Study in Programming Interdependent Activities in an Integrated Oil Company", Econometrica, Vol. 20, No. 2, April, 1952, pp. 135-159. But unlike the aviation gasoline system, the present one includes ethyl fluid concentration levels as a specific variable with a curvilinear effect.
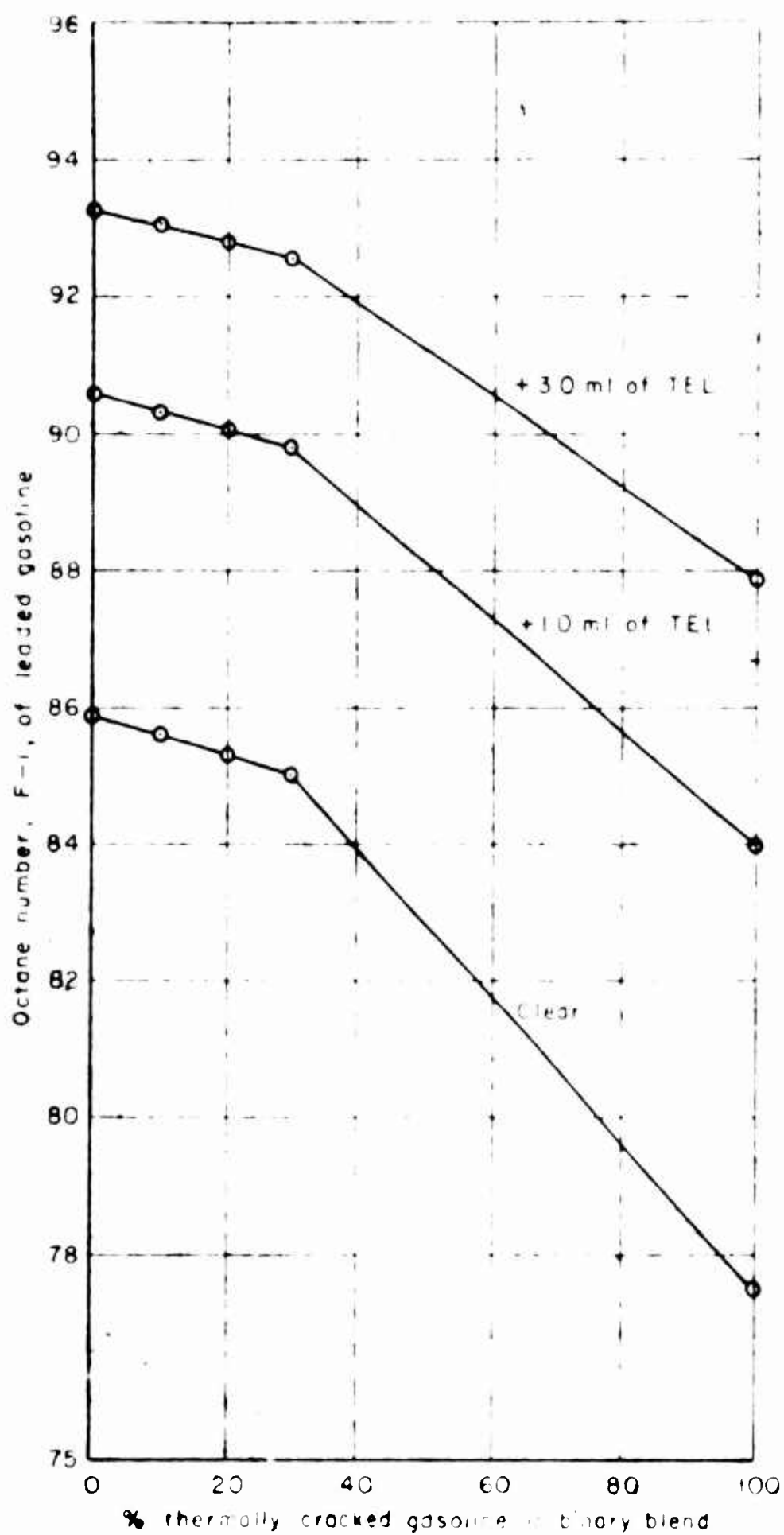
Fig.11—Octane number versus percent of thermally
cracked component in binary blend. Three TEL levels

currently at hand made it possible for the analysis to take account of non-linear TEL relationships, but not of the non-linear gasoline blending features.

The TEL inputs are one of the elements in the Union Oil Company problem that involved a departure from a strictly linear system. In addition, the cutting temperatures on two of the gasoline distillation units introduced non-linear effects. For purposes of blending, just as in the Kellogg Company naphtha reforming problem, the cut point on each of these units is an independent variable that is subject to control. The cutting temperature influences, not only the absolute quantity of material that is available for motor fuel blending, but also the quality of this component. Figures 12a - 12c contain the necessary data for one of these two distillation units -- the one producing gasoline stock #31.

According to Figure 12a, as the cutting temperature moves from 300° to 400° F., the amount of material available for motor fuel blending rises steadily from 1,900 B/CD to 3,050 B/CD. (For each additional barrel of motor fuel obtained, there is approximately a one-barrel decrease in the amount of gasoline cutter stock available.)

Charts 12.b and 12.c contain   the original Union Oil Company   curves,
also indicate the form of
and the equations that were fitted to these data. As the make of motor
                                        virtually
fuel increases, Figure 12.b indicates a linear decline in the octane number -- both clear and with 3 ml. of TEL/gallon. Also, as the make of motor fuel increases, there is a steady linear decrease in the volatility index, and a quadratically increasing sulfur content.[11]

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11. Although when distilling this particular stock the sulfur content increases quadratically, on the other material (gasoline #36) the sulfur content only changes linearly.
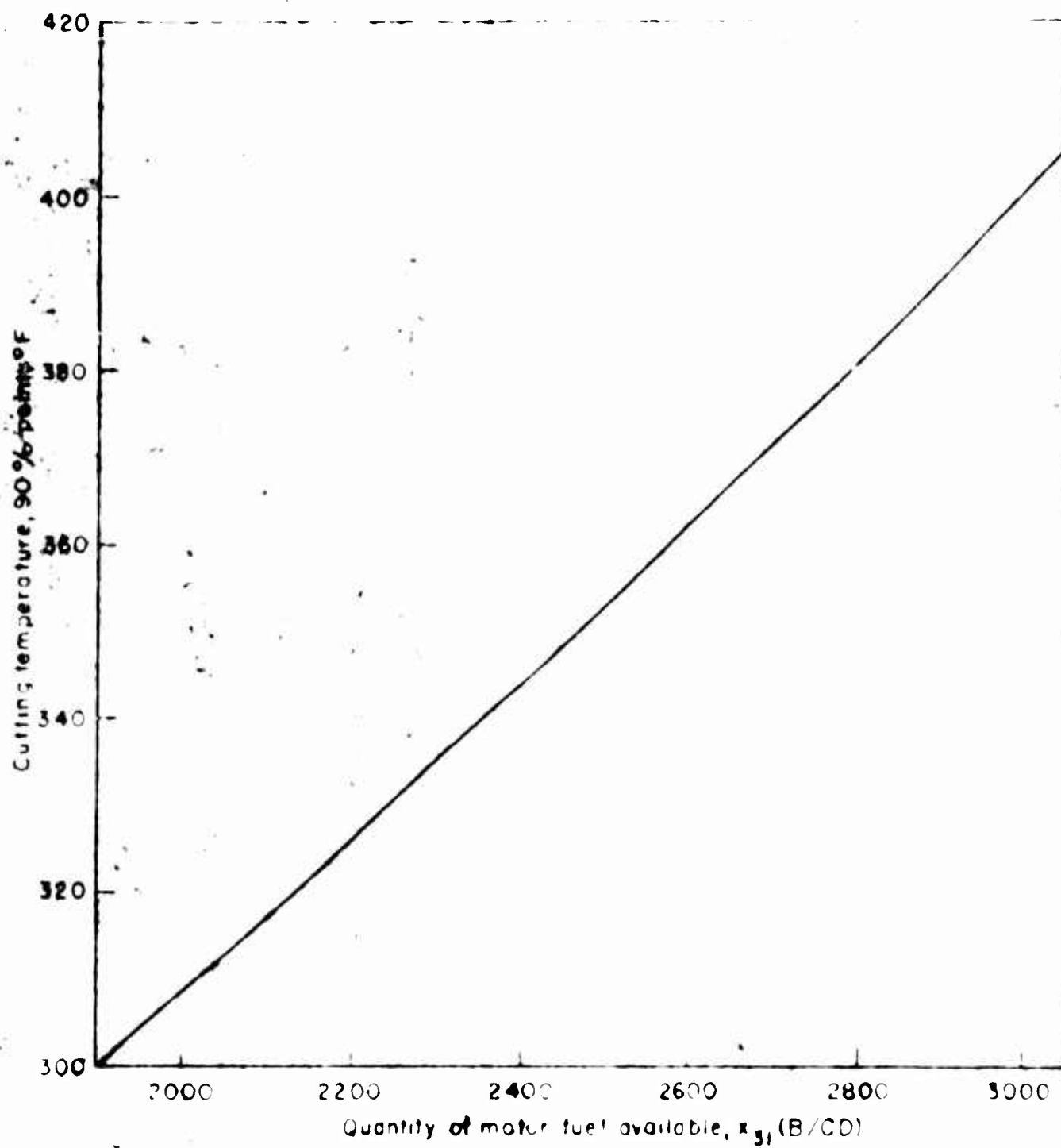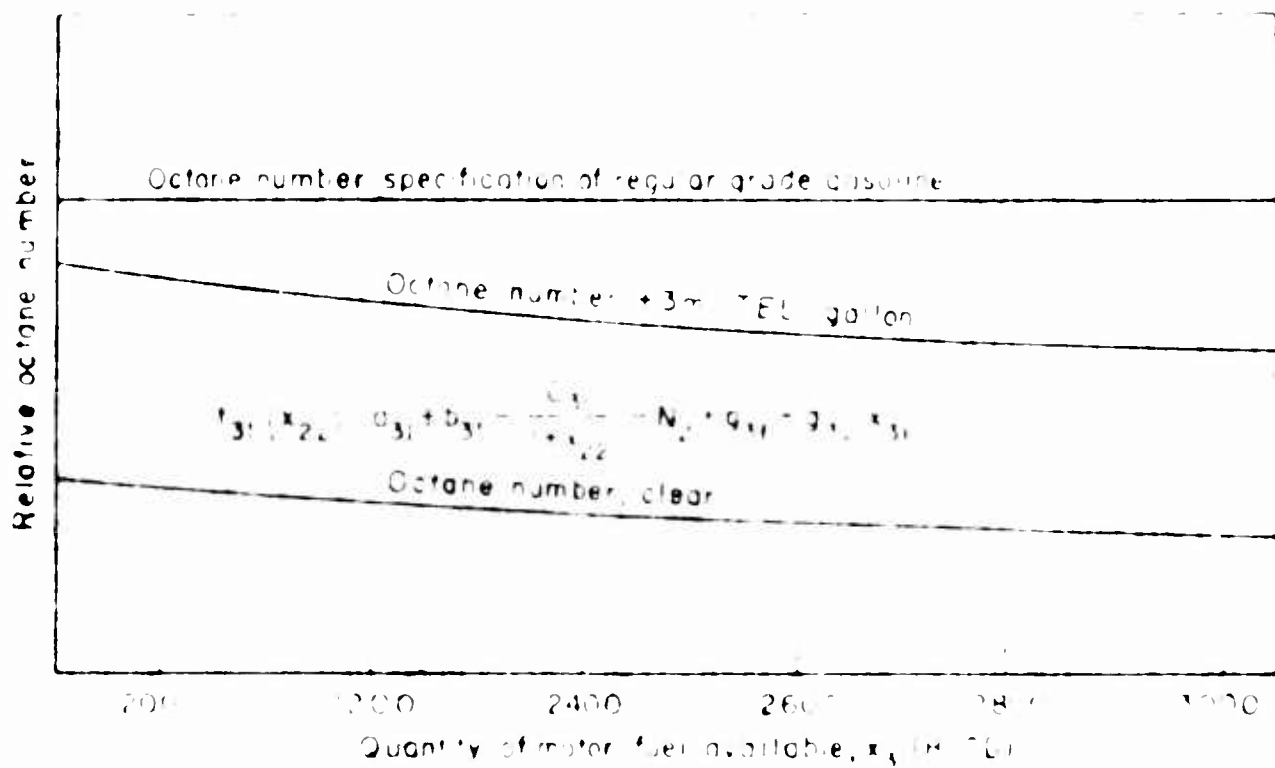
Fig. 12 a — Cutting temperature vs x_3

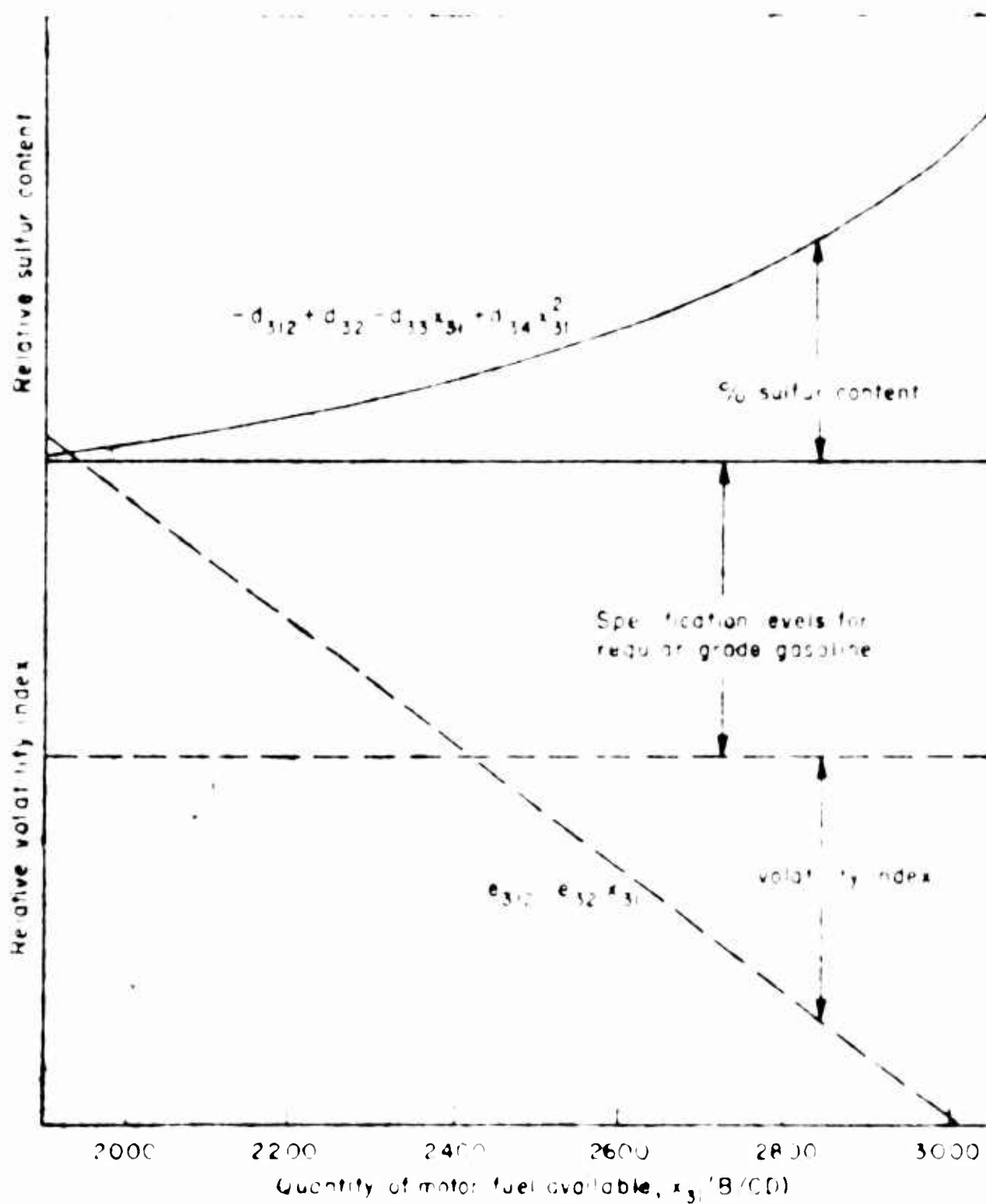Fig 12 b — Clear and leaded octane numbers vs x_{3i}

Fig. 12 c — Percent of sulfur content and volatility index vs x_3

3. The mathematical model

3.1 The basic form

Kuhn and Tucker have proved the following "equivalence theorem":

"Let the functions $f_1(x), \ldots f_m(x), g(x)$ be concave

as well as differentiable for $x \geq 0$. Then x^0 is a

solution of the maximum problem if, and only if, x^0

and some u^0 give a solution of the saddle value problem

for $\emptyset(x,u) \equiv g(x) + u'Fx.$"[12, 13]

In other words, they have established that the solution to a

concave constrained-maximum problem is equivalent to a certain mini-

max saddlepoint. At the suggestion of Harry Markowitz,   the

gasoline blending model was set up in the appropriate minmax form.

An iterative digital technique was then devised for solving the problem.[14]

From a rigorous mathematical standpoint, the present computing procedure
is an unsatisfactory one, but the

                                     method does   lead

to solutions that should be sufficiently accurate from the standpoint

of refinery managers. Convergence takes place without an inordinate

amount of computing time. All calculations were performed on a machine

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[12] The expression $u'Fx$ is shorthand for the following:
$u_1 f_1(x) + u_2 f_2(x) + \ldots + u_m f_m(x)$. The u_i are non-negative
Lagrangean multipliers, and correspond closely to the economist's
notion of "shadow prices".

[13] H. W. Kuhn and A. W. Tucker, "Nonlinear Programming", Proceedings of
the Second Berkeley Symposium on Mathematical Statistics and Probability,
ed. J. Neyman, University of California Press, 1951, p. 486.

[14] Markowitz has experimented with an electric analogue solution for this
class of applications. The present blending model involving 19 gasoline
stocks would clearly swamp the capacity of any of the present generation
of analogue machines.

that is readily available throughout the United States -- an IBM
Card Programmed Calculator, Model 2.

Dr. Martin Beckman has objected to the application of the Kuhn-
Tucker theorem in this particular instance. He has made the point
that for a function of many variables, $f(x)$, to be truly concave, it
is necessary but _not_ sufficient for that function to be concave in
each variable taken separately. It has been possible to devise ex-
amples that satisfy all conditions of the gasoline blending problem --
that is, concave in each variable taken separately -- but which never-
theless violate the strict Kuhn-Tucker conditions.[15] The counter-

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15. The following example is not derived from any Union Oil Company data,
and serves only for illustrative purposes. Consider two blending
stocks, labelled "A" and "B" with the following characteristics:

	Stock A	Stock B
Octane number, clear	74.0	90.0
Octane number, with 3.0 ml. of TEL	90.0	94.8
Number of ml./gal. required for 90 octane leaded gasoline	3.0	0

Assuming straight line interpolation at constant lead levels for the
two stocks, and employing the Hebl-Rendel-Garton blending chart, it
will be found that a 50-50 mixture would require 1.6 -- and _not_ 1.5 --
ml./gal. in order to reach the 90.0 octane level. This clearly consti-
tutes a violation of the concavity assumption, but a refiner will also
recognize that the example requires an unusual configuration of octane
numbers and lead susceptibilities.

examples give rise to multiple isolated maxima. $_\wedge$the concavity

with ...

property in the various functions, $_\wedge$to adopt blending

it is possible

schedules that are locally optimal, but which do not, in fact, repre-

sent the best of all solutions available.

The problem cannot be dismissed lightly, but I suspect that in

most particular applications such local extrema can be detected by

the investigator. The difficulties arising out of non-concavity are

not unique to the programming method described here. The same argument

would apply if the TEL levels were taken as parameters, and a separate

linear programming problem then run off for each of many combinations

of the two. This difficulty would come up in the case of any trial-

and-error method that consisted in changing one variable at a time,

and observing the effects upon the payoff. And finally, the failure

of the proper curvature conditions could frustrate optimizations by

way of a Lange-Lerner shadow price market mechanism.

3.2 Definition of independent variables

It now remains to set up the formal relationships corresponding

to the Kuhn-Tucker payoff function, $g(x)$, and the set of restraints,

$f_i(x)$. Table V.1 presents schematically the 23 independent non-nega-

tive variables, x_j, which must be selected in order to determine the

blending schedule. The whole set of 23 will be termed the vector (x).

There are 1 raw gasoline stocks, labelled 31, 3$\cdot$, 41, ... 80, as

indicated by the row headings. In each instance, we are given q_j, the

total quantity of this material available for blending. The problem

consists of assigning various amounts of the raw gasolines among the

Table V.1 Gasoline Blending Problem -- Definition of
Various Quantities

| Stocks | | | Quantity of stock j assigned to three products, B/CD | | |
| Identification No. | Quantities Available | | | | |
q_j		B/CD	(1) 7600 premium motor fuel	(2) 76 regular motor fuel	(3) Cutter Stock
31	q_{31}	3,050	0	x_{31}	$(q_{31}-x_{31})$
36	q_{36}	2,060	0	x_{36}	$(q_{36}-x_{36})$
41	q_{41}	1,000	$(q_{41}-x_{41})$	x_{41}	0
42	q_{42}	2,910	x_{45}	x_{42}	$(q_{42}-x_{42}-x_{45})$
43	q_{43}	3,460	x_{46}	x_{43}	$(q_{43}-x_{43}-x_{46})$
44	q_{44}	1,730	0	x_{44}	$(q_{44}-x_{44})$
51	q_{51}	110	$(q_{51}-x_{51})$	x_{51}	0
52	q_{52}	1,240	x_{55}	x_{52}	$(q_{52}-x_{52}-x_{55})$
53	q_{53}	860	x_{56}	x_{53}	$(q_{53}-x_{53}-x_{56})$
54	q_{54}	490	0	x_{54}	$(q_{54}-x_{54})$
60	q_{60}	6,150	$(q_{60}-x_{60})$	x_{60}	0
61	q_{61}	3,430	$(q_{61}-x_{61})$	x_{61}	0
62	q_{62}	1,460	$(q_{62}-x_{62})$	x_{62}	0
63	q_{63}	3,200	$(q_{63}-x_{63})$	x_{63}	0
64	q_{64}	1,680	$(q_{64}-x_{64})$	x_{64}	0
65	q_{65}	11,230	$(q_{65}-x_{65})$	x_{65}	0
66	q_{66}	7,440	$(q_{66}-x_{66})$	x_{66}	0
70	q_{70}	1,700	$(q_{70}-x_{70}) \equiv q_{70}$	$x_{70} \equiv 0$	0
80	q_{80}	8,710	$(q_{80}-x_{80}) \equiv 0$	$x_{80} \equiv q_{80}$	0
Totals	—	61,910	z_1	z_2	—
	TEL concentration, (ml./gal.)		x_{21}	x_{22}	—

three alternative uses -- (1) 7600, premium grade motor fuel, (2) 76, regular grade motor fuel, and (3) cutter stock. In addition, it is necessary to select x_{21} and x_{22}, respectively, the ethyl fluid con- centrations in premium and regular grade gasoline.

The computations were <u>not</u> set up with the full generality of considering $3(19) = 57$ individual possible gasoline components. Instead, for purposes of economizing on machine capacity, it was
agreed $_\wedge^{to}$ make advance guesses at which of the elements would turn out to have zero values. (Wherever a component was pre- assigned in this way, there is a zero entry in the corresponding box of Table V.1.) It was also agreed that it would not be economical to throw away any of these particular gasoline stocks.[16]   In this manner, it was possible to reduce the number of independent gasoline components from 57 down to 21.[17]

The Lagrangean multipliers, u_i, taken directly from the basic calculations, enabled us to check up on these particular assumptions at a later stage in the analysis. Fortunately, the initial guesses turned out to be correct ones. Despite the good luck for the particular sets of parameters studied, these assumptions were an unsatisfactory short-cut, and can be justified only on grounds of computational con- venience.

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[16] Even if a material could not be employed directly as cutter or as regular grade gasoline, under the 1953 price structure, it would always pay to degrade some finished 7600 gasoline for purposes of blending the particular material up to the specifications for regular grade gasoline.

[17] For symmetry in the subsequent work, there are two additional constants that are written <u>as if</u> they were independent variables. The term x_{10} is identically zero, and x_{30} identically equals q_{max}.

Table V.1 always refers to the quantity of stock j in regular grade gasoline as x_j. Even in the case of stocks #31 and #36, where the cutting temperature can equally well be regarded as the independent variable, this same convention was employed. (On Figure 12.a. note the one-to-one relationship between the cut point and the quantity of motor fuel available.)

To simplify further reference, two new dependent variables are also defined:

$$(V.2) \qquad z_1 = \sum_j (q_j - x_j) + \sum x_k \qquad \text{= total production rate, B/CD, for premium grade gasoline}$$

$$(j = 41, 51, 60-66, 70)$$
$$(k = 45, 46, 55, 56)$$

$$(V.3) \qquad z_2 = \sum_j x_j \qquad \text{= total production rate, B/CD, for regular grade gasoline}$$

$$(j = 31, 36, 41-44, 51-54, 60-66, 86)$$

3.3 Definition of the payoff function and restraints

The payoff, $g(x)$, is determined by crediting the gross refinery realization from the sale of premium and regular grade gasoline, and then charging up the ethyl fluid costs and the loss of potential cutter credits associated with the particular blending schedule:

$$(V.4) \qquad g(x) = (p_1 - p_{20} x_{21}) z_1 + (p_2 - p_{20} x_{22}) z_2$$
$$- p_{42} x_{45} - p_{52} x_{55} - p_{53} x_{56}$$
$$- \sum_j r_j x_j$$

$$(j = 31, 36, 42, 43, 44, 52, 53, 54)$$

In equation (V.4), the various unit costs and prices are defined as follows:

(V.5) p_1 = refinery realization on 100, premium gasoline, $/B

p_2 = refinery realization on 76, regular grade gasoline, $/B

p_{20} = cost of ethyl fluid, $/42 ml.

p_j = unit cutter credit on stock j, $/B

(j = 31, 36, 42, 43, 44, 52, 53, 54)

The blending schedule (x) must not call for the production of negative amounts of any motor fuel or cutter component, and so there are certain upper and lower limits imposed upon the various x_j. In addition, the schedule must not require TEL concentration levels exceeding 3.00 ml./gallon. And because of the flash point specification on finished No. 6 oil, it must not require cutter components #31 or #36 to fall below q_{31}^o and q_{36}^o, respectively.

(V.6) $0 \leq x_{21} \leq 3.0$

$0 \leq x_{22} \leq 3.0$

$q_{31}^o \leq x_{31} \leq q_{31}$

$q_{36}^o \leq x_{36} \leq q_{36}$

$0 \leq x_i \leq 4_i$

$0 \leq x_j$

(i = 41, 44, 51, 54, 60-66)

(j = 42, 43, 45, 46, 52, 53, 55, 56)

For stocks #42, 43, 52, and 53, the   quantities of individual

cutter stocks, $f_i(x)$, must not be negative:

(V.7) $f_1(x) = q_{42} - x_{42} - x_{45} \geq 0$

$f_2(x) = q_{43} - x_{43} - x_{46} \geq 0$

$f_3(x) = q_{52} - x_{52} - x_{55} \geq 0$

$f_4(x) = q_{53} - x_{53} - x_{56} \geq 0$

The six remaining equations are connected with the three specifica-
tions on each of the two motor fuel products. The functions $f_5(x)$ and
$f_6(x)$ relate, respectively, to the octane number of premium and regular
grade motor fuel; $f_7(x)$ and $f_8(x)$ to the sulfur content; and $f_9(x)$ and
$f_{10}(x)$ to the volatility index. These relationships are all defined
in such a way that the particular specification will be satisfied if,
and only if, the function $f_i(x) \geq 0$. To simplify notation, the constants
in the equations do not refer to the absolute octane number, sulfur con-
tent, or volatility index of a particular raw gasoline, but rather to
the differences between these values and those of the specifications
for a particular motor fuel product.

In the case of the octane ratings, for example, equation (V.1)
indicates the relationship between the octane number of a particular
blending stock, and the TEL concentration in a finished product. A
new function, $t_{jk}(x_{2k})$ is defined in order to indicate the number of
octane points by which stock j exceeds the specifications for product
$\underline{k}$, when x_{2k} ml. of lead are added to stock j. The variable t_{jk} is
evidently a function of x_{2k}, and is determined through the following
relation:

$$(V.8) \qquad t_{jk}(x_{2k}) = a_j + b_j x_{2j} - \frac{c_j}{1+x_{2k}} - N_k \qquad (k = 1,2)$$

The terms a_j, b_j, and c_j represent the lead susceptibility constants for stock j, and correspond to the three constants in equation (V.1). The parameter N_k stands for the minimum octane number requirement connected with gasoline blend $\underline{k}$.

In order to ensure meeting octane number specifications on the premium grade gasoline, the following condition must then apply:

$$(V.9) \qquad f_5(x) = \sum_j t_{j1}(q_j - x_j) + \sum_i t_{i1}x_i \geq 0 \qquad \begin{array}{l}) \; j = 41, 51, \\ ) \quad 60\text{-}66, 70 \\ ) \; i = 45, 46, \\ ) \quad 55, 56 \end{array}$$

Similarly, to make sure that the regular grade gasoline blend will pass its octane number specification, we must have:

$$(V.10) \qquad \begin{array}{l} f_6(x) = \sum_j t_{j2}x_j + x_{31}\left[g_{31} - g_{32}x_{31}\right] \\ \qquad + x_{36}\left[g_{36} - g_{37}x_{36}\right] \geq 0 \end{array} \qquad \begin{array}{l}) \; j = 31, 36, \\ ) \quad 41\text{-}44, \\ ) \quad 51\text{-}54, \\ ) \quad 60\text{-}66, \\ ) \quad 80 \end{array}$$

The functions in brackets that follow the variables x_{31} and x_{36} are inserted in order to reflect the fact that with these two gasolines, as the cutting temperature changes, both the quantity and the octane number of the motor fuel component will be affected. (See Figure 12.b.) This feature evidently leads to negative square terms involving x_{31} and x_{36}.

The same two stocks give rise to quadratic expressions in connection with the sulfur and volatility specifications for regular grade gasoline, but since these components were assumed to be absent from the premium gasoline blend, those equations remain completely linear. For the sulfur specification on the premium grade gasoline, the condition reads:

$$(\text{v.11}) \qquad f_7(x) = \sum_j d_{j1}(q_j - x_j) + \sum_k d_{k1}x_k \geq 0$$

$$j = 41, 51, 60\text{-}66, 70$$

$$k = 45, 46, 55, 56$$

The constant term d_{j1} represents the algebraic excess of the maximum allowable sulfur content of 7600 gasoline over the amount actually in stock j. There is a similar set of constants associated with the regular grade gasoline -- d_{j2}, the difference between the allowable sulfur content in the 76 grade product and the content in stock j. The following relationship controls the sulfur level in the finished regular grade product:

$$(\text{v.12}) \qquad f_8(x) = \sum_j d_{j2}x_j - x_{31}[d_{32} - d_{33}x_{31} + d_{34}x_{31}^2]$$

$$- x_{36}[d_{37}x_{36}] \geq 0$$

$$j = 31, 36, 41\text{-}44, 51\text{-}54, 60\text{-}66, 80$$

The bracketed terms following x_{31} and x_{36} arise out of the cutting temperature relationship. (See Figure 12. c.) Note that over the entire range between q_{31}^o and q_{31}, the terms involving x_{31} bring about a strictly concave function. That is, the following partial second derivative is negative over this particular interval:

$$(\text{v.13}) \qquad \frac{\partial^2 f_8(x)}{\partial x_{31}^2} = 2d_{33} - 6d_{34}x_{31} < 0$$

The functions $f_9(x)$ and $f_{10}(x)$ do the job, respectively, of governing the volatility of the premium and regular grade gasoline:

$$(V.14) \qquad f_9(x) = \sum_j e_{j1}(q_j - x_j) + \sum_k e_{k1}x_k \geq 0$$

$$j = 41, 51, 60\text{-}66, 70$$

$$k = 45, 46, 55, 56$$

$$(V.15) \qquad f_{10}(x) = \sum_j e_{j2}x_j - e_{32}x_{31}^2 - e_{37}x_{36}^2 \geq 0$$

$$j = 31, 36, 41\text{-}44, 51\text{-}54, 60\text{-}66, 80$$

Here again, because of the cutting temperature-quality relationship, there are square terms connected with x_{31} and x_{36}.

The gasoline blending model has now been reduced to a constrained-maximum mathematical form. The problem consists of choosing a vector (x) that will maximize $g(x)$, subject to the side conditions that $f_i(x) \geq 0$, and also subject to certain upper and lower limits on the individual components, x_j. The payoff function $g(x)$ is given by equation $(V.4)$. The upper and lower limits on individual variables are stated in $(V.6)$, and the 10 restraining inequalities, $f_i(x)$, in $(V.7)$, $(V.9)$, $(V.10)$, $(V.11)$, $(V.12)$, $(V.14)$, and $(V.15)$.

In this form, the problem _almost_ fits the Kuhn-Tucker conditions. Both the payoff function and the restraints are concave in each variable taken separately.[18] Furthermore, eight of the ten restraints are concave over the whole space -- $f_1(x) \ldots f_4(x)$, and $f_7(x) \ldots f_{10}(x)$.

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[18] Kuhn and Tucker give the following definition of concavity: "A function is _concave_ if linear interpolation between its values at any two points of _definition_ yields a value not greater than its actual value at the point of interpolation". _Op. cit._, p. 481. The function is said to be concave in each variable _separately_ if the two points of definition are always chosen so that only one of the components of the vector (x) is altered. A function may be concave in each variable separately _without_ being concave over the whole space of points (x).

The payoff function $g(x)$ and the two octane number restraints -- $f_5(x)$ and $f_6(x)$ -- do not strictly satisfy the full concavity properties. It was decided, nevertheless, to perform the computations as if these also satisfied the rigorous Kuhn-Tucker conditions. Under this approach, there is always the danger that a local optimum will not turn out to be the <u>maximum</u> <u>maximorum</u>. A number of alternative computing schemes were examined, but all of these suffered from precisely the same defect. The primary justification for going ahead in this non-rigorous fashion was that the computational results could always be checked against the experience of the refinery engineers. It was felt that these individuals would certainly be able to detect any errors of a gross nature.

4. <u>The numerical analysis</u>

4.1 Outline of computational procedure

Following the assumption that this gasoline blending model fits into the format of the Kuhn-Tucker theorem, it is necessary to define the Lagrangean function $\emptyset(x,u)$:

$$(V.16) \qquad \emptyset(x,u) = g(x) + \sum_{i=1}^{10} u_i f_i(x)$$

The constrained-maximum problem will be solved if, and only if, there is a saddlepoint solution to the function, $\emptyset(x,u)$. The computational procedure is an iterative one — at each step $\underline{t}$, converting a vector $x(t)$, $u(t)$ into a new vector $x(t+1)$ and $u(t+1)$. The solutions observed have all tended toward a saddlepoint, but I can give no strict proof of the necessity of this convergence. The new vector generated is never exactly "efficient", nor is it "attainable" in the sense of Koopmans.[19] The payoff does not increase monotonically as in each successive step of Dantzig's simplex procedure for linear programming.[20] Despite these apparent shortcomings, the algorithm has given useful answers in the cases examined to date.

The solution must be started off from some initial point $x(0)$, $u(0)$. In principle, this may be any arbitrary non-negative vector. In practice, though, it is possible to effect a considerable reduction in computing time if a good initial position is selected. At each step, first the (u) vector is determined, and then the (x) vector. For the former, the

19. T.C. Koopmans, "Analysis of Production as an Efficient Combination of Activities", Chapter III, <u>Activity Analysis of Production and Allocation</u>, p. 79.

20. G.B. Dantzig, "Maximisation of a Linear Function of Variables Subject to Linear Inequalities", <u>ibid.,</u> Chapter XXI, pp. 339-347.

basic iteration consists of two steps, and for the latter, of four

steps. The problem is one of determining $\Delta u_1 \equiv u_1(t + 1) - u_1(t)$, and

$\Delta x_j \equiv x_j(t + 1) - x_j(t)$. The procedure appears cumbersome, but only

six minutes of CPC machine time are actually required for generating

a whole set of Δu_1 and Δx_j. (The required total number of steps has

varied between 50 and 80.) For the Δu_1, the procedure is as follows:

   1. If $\dfrac{\partial \phi(t)}{\partial u_1(t)} \geq 0$, then the "candidate" $\widehat{\Delta u_1} = -k_1$

      If $\dfrac{\partial \phi(t)}{\partial u_1(t)} < 0$, then the "candidate" $\widehat{\Delta u_1} = k_1$

Note: k_1 is an arbitrary positive constant.

   2. If $u_1(t) + \widehat{\Delta u_1} < 0$, then $\Delta u_1 = -u_1(t)$

      If $u_1(t) + \widehat{\Delta u_1} \geq 0$, then $\Delta u_1 = \widehat{\Delta u_1}(t)$

For the Δx_j, the procedure becomes:[21]

   1. If, for any i, $f_i(t) < 0$, **and** if $\left[\dfrac{\partial \phi(t)}{\partial x_j(t)} \cdot \dfrac{\partial f_i(t)}{\partial x_j(t)}\right] < 0$, then

      $\Delta x_j = 0$; proceed to evaluate Δx_{j+1}. Otherwise, proceed

to step 2.

   2. If $\dfrac{\partial \phi(t)}{\partial x_j(t)} \geq 0$, then the "candidate" $\widehat{\Delta x_j} = k_j$

      If $\dfrac{\partial \phi(t)}{\partial x_j(t)} < 0$, then the "candidate" $\widehat{\Delta x_j} = -k_j$

Note: k_j is an arbitrary positive constant.

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[21] The individual Δx_j are determined in ascending order of their respective indices, j. To simplify the notation, Δx_{j+1} is used to indicate the next Δx_j that is to be determined. For example, Δx_{36} follows Δx_{31} in sequence, and for $j = 31$, is indicated by Δx_{j+1}.

3. a. If $j = 31$ or 36, <u>and</u> if $x_j(t) + \Delta \hat{x}_j < q_j^0$, then

   $\Delta x_j = x_j(t) - q_j^0$; proceed to evaluate Δx_{j+1}.

   If $j = 31$ or 36, <u>and</u> if $x_j(t) + \Delta \hat{x}_j \geq q_j^0$, then proceed

   to step 4.

   b. If $j \neq 31$ or 36, and if $x_j(t) + \Delta \hat{x}_j < 0$, then

   $\Delta x_j = -x_j(t)$; proceed to evaluate Δx_{j+1}.

   If $j \neq 31$ or 36, and if $x_j(t) + \Delta \hat{x}_j \geq 0$, then proceed

   to step 4.
   $$\Delta x_j = \Delta \hat{x}_j;$$

4. a. If $j = 42, 43, 45, 46, 52, 53, 55$ or 56, proceed to

   evaluate Δx_{j+1}.

   b. If $j = 21$ or 22, calculate $[3.00 - x_j(t) - \Delta \hat{x}_j]$.
   If this expression is non-negative, $\Delta x_j = \Delta \hat{x}_j$.
   Otherwise $\Delta x_j = 3.00 - x_j(t)$. Proceed next to
   evaluate Δx_{j+1}.

   c. If $j = 41, 44, 51, 54$, or $60, 6.,\ \ldots 66$, calculate
   $[q_j - x_j(t) - \Delta \hat{x}_j]$. If this expression is non-
   negative, $\Delta x_j = \Delta \hat{x}_j$. Otherwise $\Delta x_j = q_j - x_j(t)$.
   Proceed next to evaluate Δx_{j+1}.

For the Δu_i, the explanation of this ritual is straightforward.
Step (1) tells us to decrease u_i by an arbitrary amount k_i if, for a
"small" change in u_i <u>alone</u>, the effect will be to decrease $\phi(x,u)$.
Similarly, there is to be an increase in u_i if, for a small increase,
ϕ would decrease. Step (2) prevents u_i from becoming negative.

For the Δx_j, the justification is more roundabout. (Steps (3) and (4) are the obvious ones -- respectively, lower and upper limits on the individual x_j dictated by (V.6).) As with the Δu_i the sign of $\frac{\partial \phi(t)}{\partial x_j(t)}$ is primarily the criterion that determines whether to take a positive or a negative step.[22] Preliminary small-scale calculations indicated, however, that the unqualified "direction of ascent (descent) rule" would lead to major oscillations rather than to convergence at equilibrium values. For that reason, rule (1), a relaxation-type principle, was inserted. Translated into English, the statement reads, "If one of the restraints is being violated, and if the direction-of-ascent rule would lead to an even greater violation of this restraint, then the particular x_j should not be changed." The relaxation rule is hardly elegant from the viewpoint of pure mathematics, and there are undoubtedly other possibilities for achieving the same effect.

At the present time, despite their plausibility, there is a serious dilemma concerning the logic of these computational techniques. On the one hand, I have been unable to prove the necessity of convergence for either the relaxation or the unmodified procedure. On the other, I have been unable to construct a counterexample for a case where the k_i and the k_j may be made arbitrarily small. Readily granting these apparent objections, the fact remains that the relaxation technique has given useful results in a series of computing runs

V.2 Details of the numerical analysis

Three sets of calculations have been performed. The first is to be considered the basic case -- taking as parameters the 1953

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[22] The reader should note that ϕ is being minimized with respect to (u), and maximized with respect to (x).

refinery realizations, product specifications, and raw material avail-
abilities. The second run was concerned with a radical change in the
price structure -- an 11.6% drop in the value of premium grade gasoline.
In the third, the price structure was taken to be the same as in
the basic case, but a 2.5 octane point increase was imposed upon the
premium product. For each of these departures from the basic $\wedge$ conditions, the
optimal blending schedule becomes considerably altered.

In order to convey some indication of the computational technique
employed here, Figures 13.a - 13.e. chart the values of certain of the
variables during the last twenty steps of Run #1, i.e., between t 40 and
t - 60. According to Figure 13.a, the payoff and the aggregate make of
premium and regular grade gasoline had become stabilized, and "rippled"
steadily about certain equilibrium levels. Virtually all of the inde-
pendent variables exhibit similar rippling around an equilibrium point,
but there were exceptions. The worst drifting occurred in the case of
x_{42} and x_{45}, respectively, the quantities of stock #42 in regular and
premium grade gasoline. Their time paths are shown in Figure 13.b.
The more typical pattern was exhibited by the two TEL concentration
variables, x_{21} and x_{22}, and is reproduced in Figure 13.c. Two of the
Lagrangean multipliers, u_5 and u_6, connected with the two octane number
specifications, are shown in Figure 13.d. The octane number functions
themselves, $f_5(x)$ and $f_6(x)$, are given in Figure 13.e.

Before cutting off any individual computation run, it was possible
to perform an additional check upon the stability of the solution.
According to the Kuhn-Tucker theorem, in order for the vector $(x)^0$ to
be a saddle value solution, the following condition must hold for all
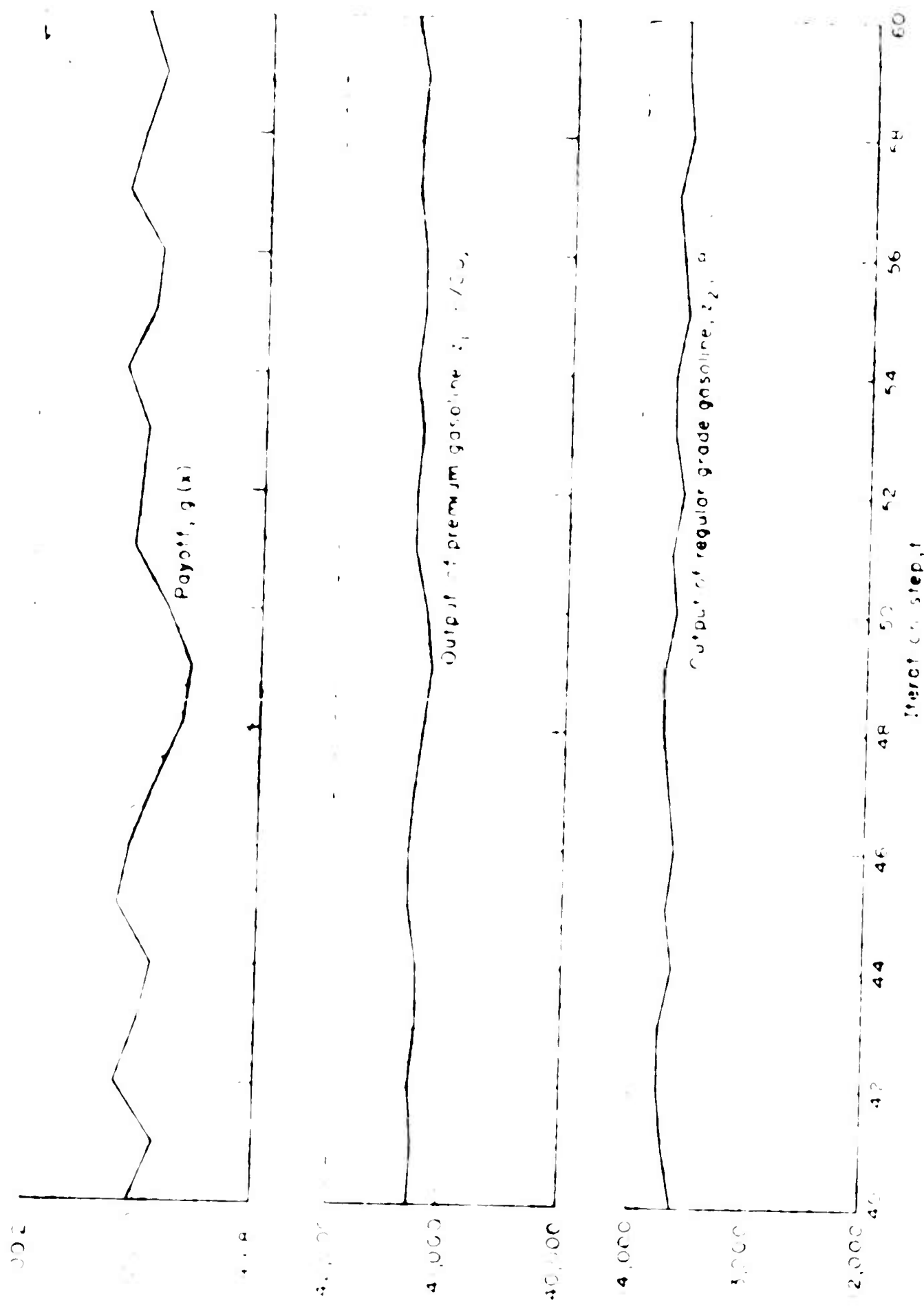x_j^0: [23]

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23. Op. cit., p. 482.

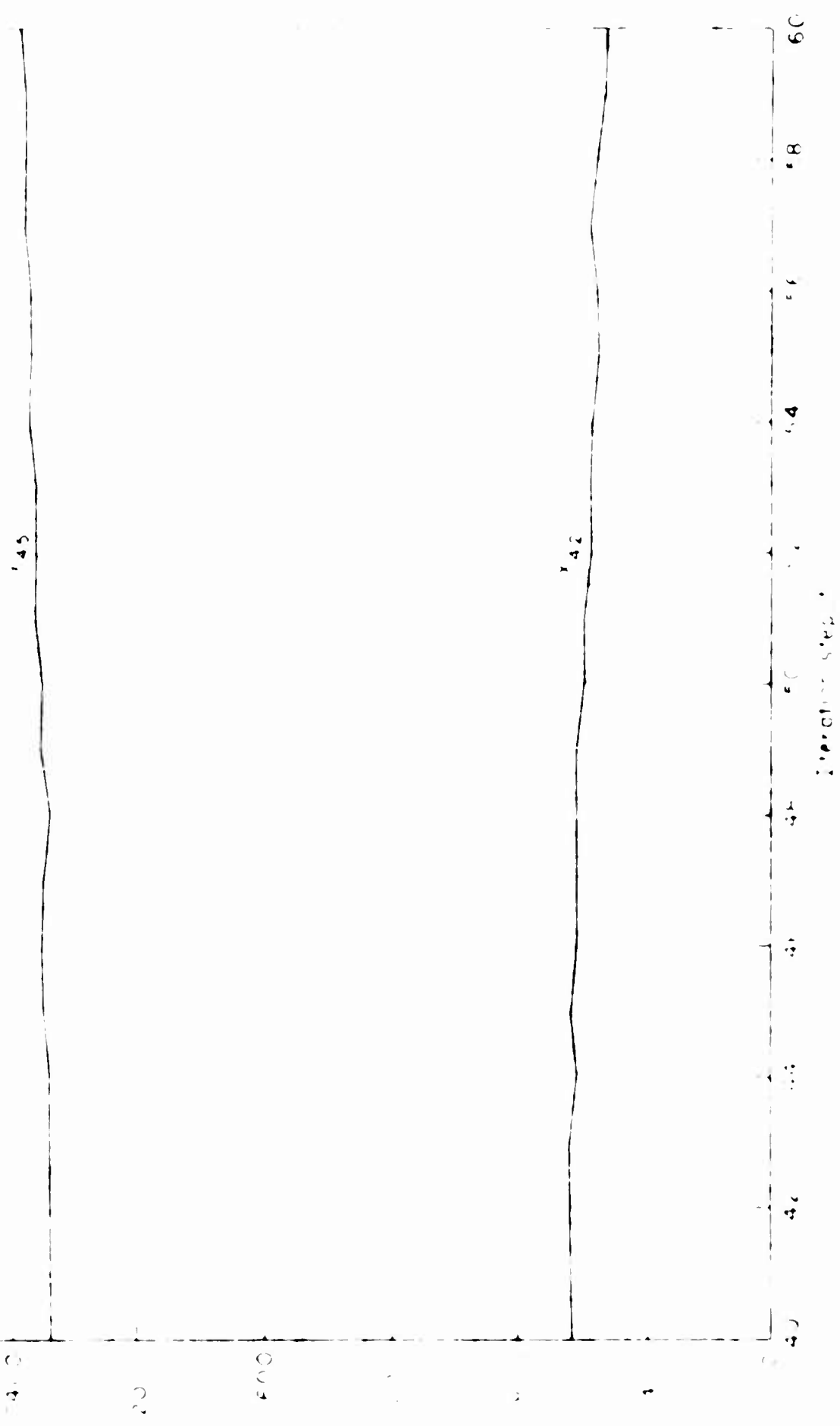Fig. 13a — Run no.1, payoff and total gasoline output

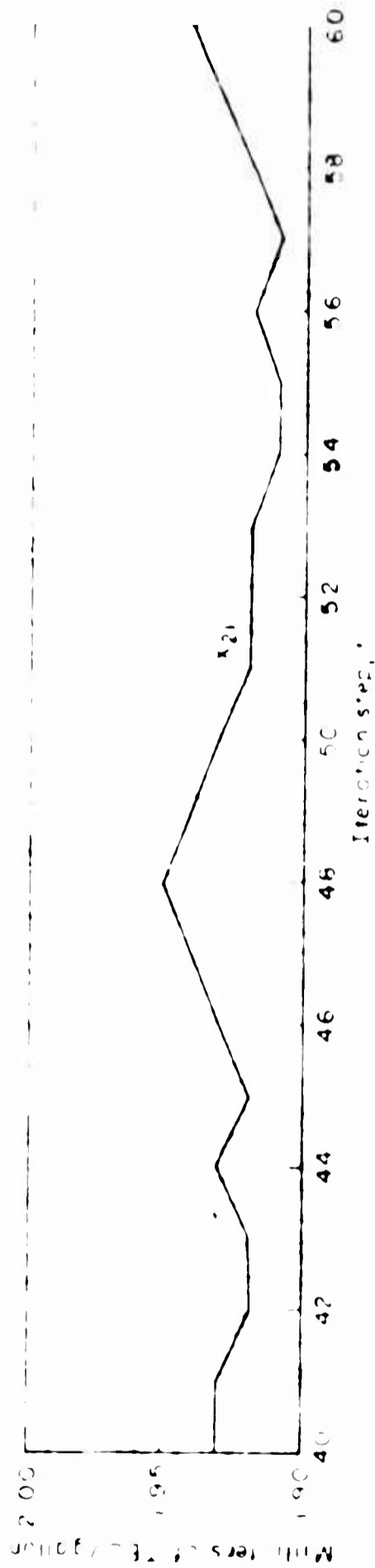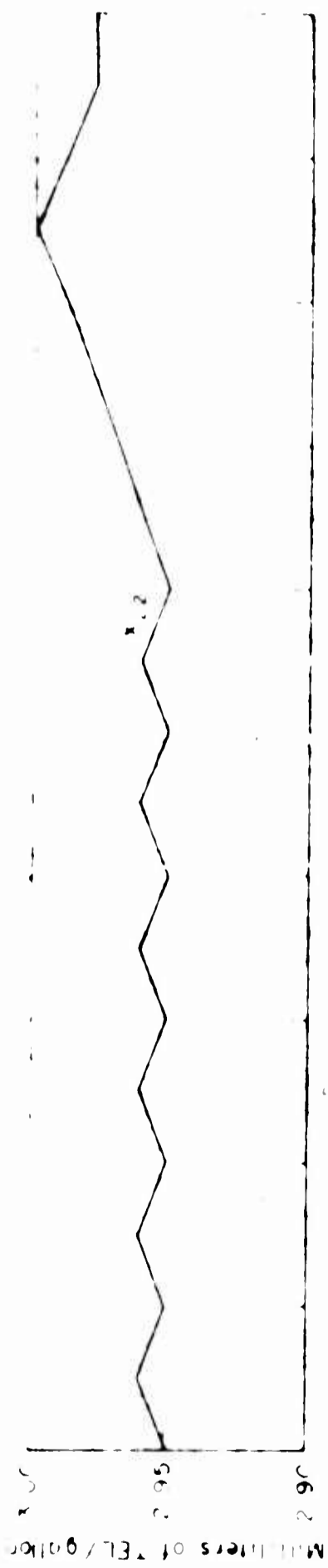Fig. 13 b — Run no. 1, gasoline components x_{42} and x_{45}

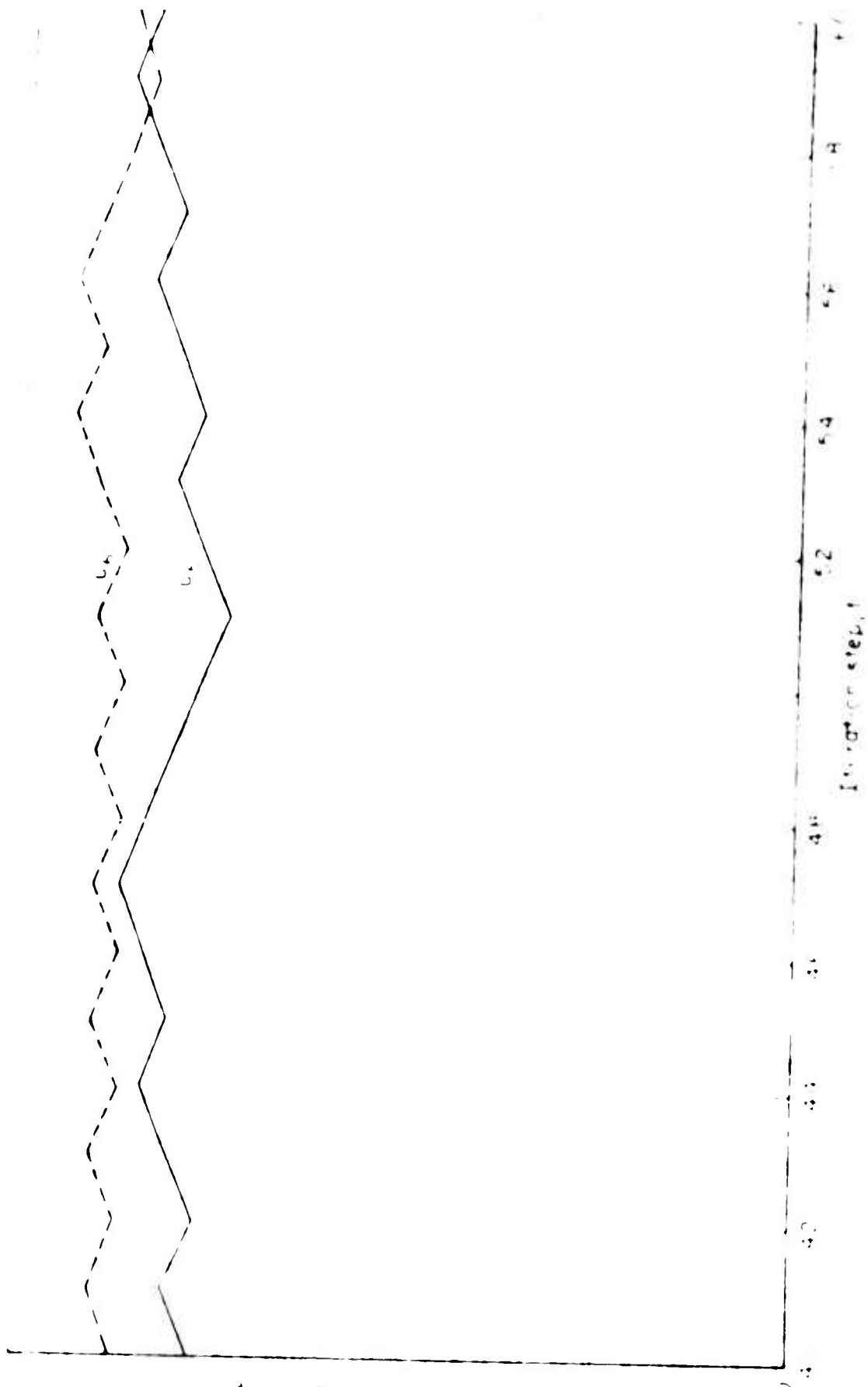Fig. 13c — Run no. I, TEL concentration levels x_{21} and x_{22}

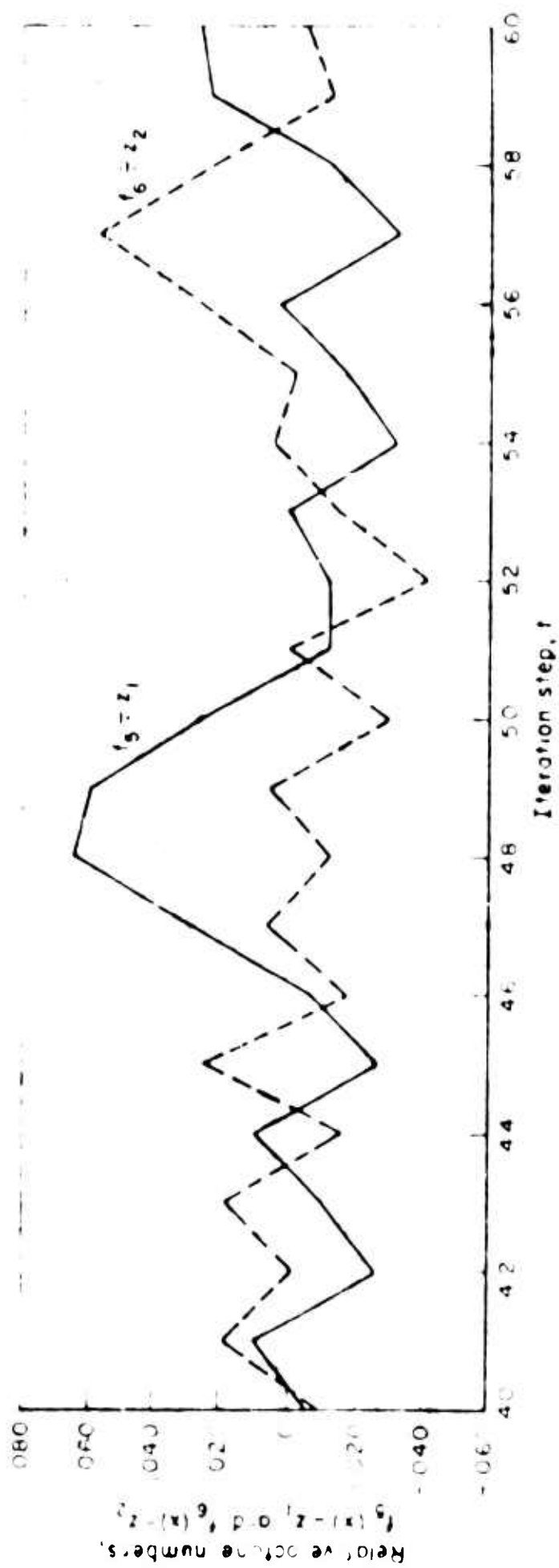Fig. 13 d — Run no.1, Lagrangean multipliers, u_5 and u_6

Fig 13e — Run no. 1, relative octane numbers, $f_5(x) \div z_1$, and $f_6(x) \div z_2$

$$(V.17) \qquad \frac{\partial \phi^0}{\partial x_j^0} \leq 0, \qquad x_j^0 \; \frac{\partial \phi^0}{\partial x_j^0} = 0, \qquad x_j^0 \geq 0$$

In the present case, where the x_j are bounded from above as well as below, this condition on the equilibrium x_j^0 becomes:[24]

$$(V.18) \qquad \text{a. If} \; \frac{\partial \phi^0}{\partial x_j^0} \leq 0, \; \text{then} \; x_j^0 \; \frac{\partial \phi^0}{\partial x_j^0} = 0, \; \text{and} \; x_j^0 \geq 0$$

$$\qquad \text{b. If} \; \frac{\partial \phi}{\partial x_j^0} > 0, \; \text{then} \; x_j^0 = q_j$$

Using condition (V.18), it is possible to perform a rough check on the precision of the solution. For those x_j that are at their lower limits, $\frac{\partial \phi^0}{\partial x_j^0} = 0$. For those that are at their upper limits, $\frac{\partial \phi^0}{\partial x_j^0} \geq 0$. And for those that are neither at their upper nor at their lower limits, $\frac{\partial \phi^0}{\partial x_j^0} = 0$. In this latter case, it is not to be expected that the partial derivative will be literally at a zero level. Rather, since this is an approximative method of solution the derivative should be in the <u>neighborhood</u> of zero.

Table V.2 contains the actual values for the $x_j(60)$ and for the normalized values of the partial derivatives $\frac{\partial \phi(60)}{\partial x_j(60)}$. All of those variables that are at their lower limits exhibit negative partial derivatives, as is to be expected. On the other hand, the single variable

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[24] In the case of (V.18a), where $j = 31$ or 36, the lower limits q_j^0 must be interpreted as equivalent to zero. For (V.18b), where $j = 21$ or 22, the upper limits of 3.00 ml. are to be regarded as the appropriate q_j. For those x_j on which there is no specific imposed upper limit -- that is, $j = 42, 43, 45, 46, 52, 53, 55,$ and 56 -- the condition (V.18b) does not apply.

Table V.2   Comparison of Partial Derivatives for t = 60, Run #1

i	Lower Limit on x_j	$x_j(60)$	Upper Limit on x_j	$\dfrac{\partial \phi(60)}{\partial x_j(60)}$		Adjusted Optimal x_j
				milliliter/gallon in premium gasoline		
21	0	1.94	3.00	1.57% of cost of 1_A		1.92
22	0	2.98	3.00	-0.61% of cost of 1_V		2.96
				milliliter/gallon in regular gasoline		
31	1,900	1,900	3,050	-22.54% of realization on 1 barrel of premium gasoline		1,900
36	1,320	1,320	2,060	-0.34 "	"	1,320
41	0	0	1,000	-5.58% of realization on 1 barrel of premium gasoline		0
42	0	540	2,910	-0.88 "	"	620
43	0	400	3,460	-0.40 "	"	380
44	0	0	1,730	-8.62 "	"	0
45	0	2,300	2,910	2.35 "	"	2,290
46	0	850	3,460	1.09 "	.	845
51	0	0	110	-4.41% of realization on 1 barrel of premium gasoline		0
52	0	0	1,240	-3.41 "	"	0
53	0	280	860	-0.05 "	"	260
54	0	4.0	490	-0.08 "	"	480
55	0	100	1,240	0.26 "	"	120
56	0	600	860	1.27 "	"	600
60	0	0	6,150	-6.90% of realization on 1 barrel of premium gasoline		0
61	0	0	3,430	-5.98 "	"	0
62	0	0	1,460	-9.82 "	"	0
63	0	0	3,200	-8.55 "	"	0
64	0	0	1,680	-9.10 "	"	0
65	0	0	11,230	-7.94 "	"	0
66	0	0	7,440	-7.35 "	"	0

that lies at its upper limit, x_{54}, does not have a partial derivative of the proper sign. This derivative is slightly negative, and in fact, during the sequence leading up to $t = 60$, $x_{54}(t)$ had oscillated between 470 B/CD and 4 0. There appears to be a coincidence between the upper limit on this particular variable, and the point at which $\frac{\partial g}{\partial x_{54}}$ vanishes.

In the case of those variables that lie neither at their upper nor their lower limits, the maximum absolute deviation from zero amounts to 2.3% of the value of one barrel of premium grade gasoline. This deviation occurred in the case of x_{45}, the variable shown in Figure V.4b that had been slowly drifting upward. It did not seem worthwhile to continue the iterations in order to refine upon this one estimate.

The blending schedule selected at step 60 is neither literally "attainable". The "efficient" nor $\wedge$ payoff $g(x)$ is below the and for level recorded for step 45, $\wedge$ this reason, the final blending schedule was taken over from that earlier step. Certain minor corrections were made by hand in order to bring about a feasible and efficient solution. The resulting vector is listed in the final column under the heading "Adjusted optimal x_j".

5. Results of the numerical analysis

5.1 Interpretation of results -- blending programs

The operations schedules derived from the three computation runs
are ʌ _presented_ in Table V.3. For reference purposes, the first three
columns indicate the program that was effective at the beginning of the
year 1953. The other sets give the results for run #1, the basic case;
run #2, a decrease of 11.58% in the realization on premium grade gasoline;
and run #3, an increase of 2.5 octane numbers in the specification for the
premium product.

The outstanding features of these different schedules are summarized
in bar charts, Figures 14.a. - 14.d. Of all four situations, the output of
7600 gasoline is largest in the case of run #1, and smallest
under "1953 operations". Just the reverse is true for the regular grade
gasoline. Its production is largest under the 1953 operations plan, and
smallest in the basic case. Going along with this considerable shift in
the general production strategy, there is an **apparently** small shift in pay-
off, for the basic case would yield a return to the company of only 3.99
percentage points in excess of that shown in actual practice.[25] At
first glance this figure of 3.99 does not seem impressive, but neverthe-
less, during the course of a single year's operation a differential of this
magnitude amounts to several millions of dollars! From the company's
standpoint, it would look eminently worthwhile to shift over to the
blending schedule derived from run #1.

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25. In order to avoid revealing proprietary information, no absolute dollar
figures are presented here. Both the unit and the total refinery
realizations are stated in terms of index numbers.

Table V.

Stocks		Operation		
Identification Number	Quantities Available. B/CD	7600		Cutter Stock
31.	3,090	—	2,260	700
32.	2,0	—	1,300	170
41.	1,000	900	90	—
42.	2,310	1,450	1,450	0
43.	3,400	0	3,400	0
44.	1,730	—	1,730	0
51.	110	55		—
52.	1,240	620	620	0
53.	840	0	840	0
54.	440		440	0
60.	6,110	3,000	3,110	—
61.	3,430	0	3,430	—
62.	1,440	0	1,440	—
63.	3,200	0	3,200	—
64.	1,680	0	1,680	—
65.	11,230	8,720	2,510	—
77.	7,440	7,040	680	—
70.	1,700	1,500		—
80.	8,710		8,710	—
Totals	61,410	22,910	33,040	700
TEL Concentration (ml./gal.)	—	1.13	1.0	—
Total Consumption of TEL (liters/day)			2,700	
Payoff, $g(x)$, percent of total for Run #1, Basic Case			9.01	
Refinery realization on motor gasoline, percent of 195 realization on 7600 product		10.20	52.4	—
Octane rating of motor gasoline product, number of points in excess of 190 housebrand specifications		0	0	—

Results of Computations

Quantity of Stock Assigned to Three Products, B/CD

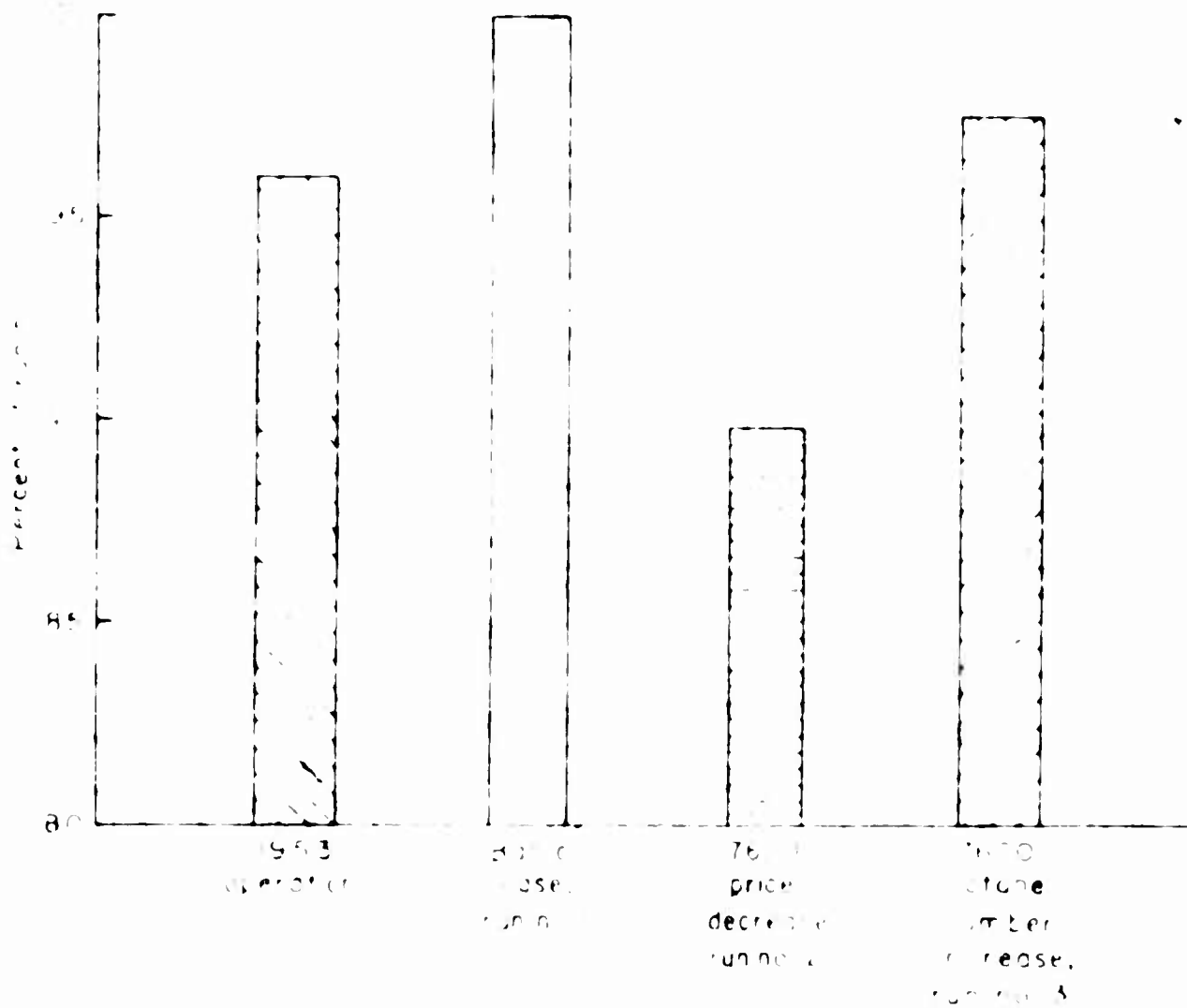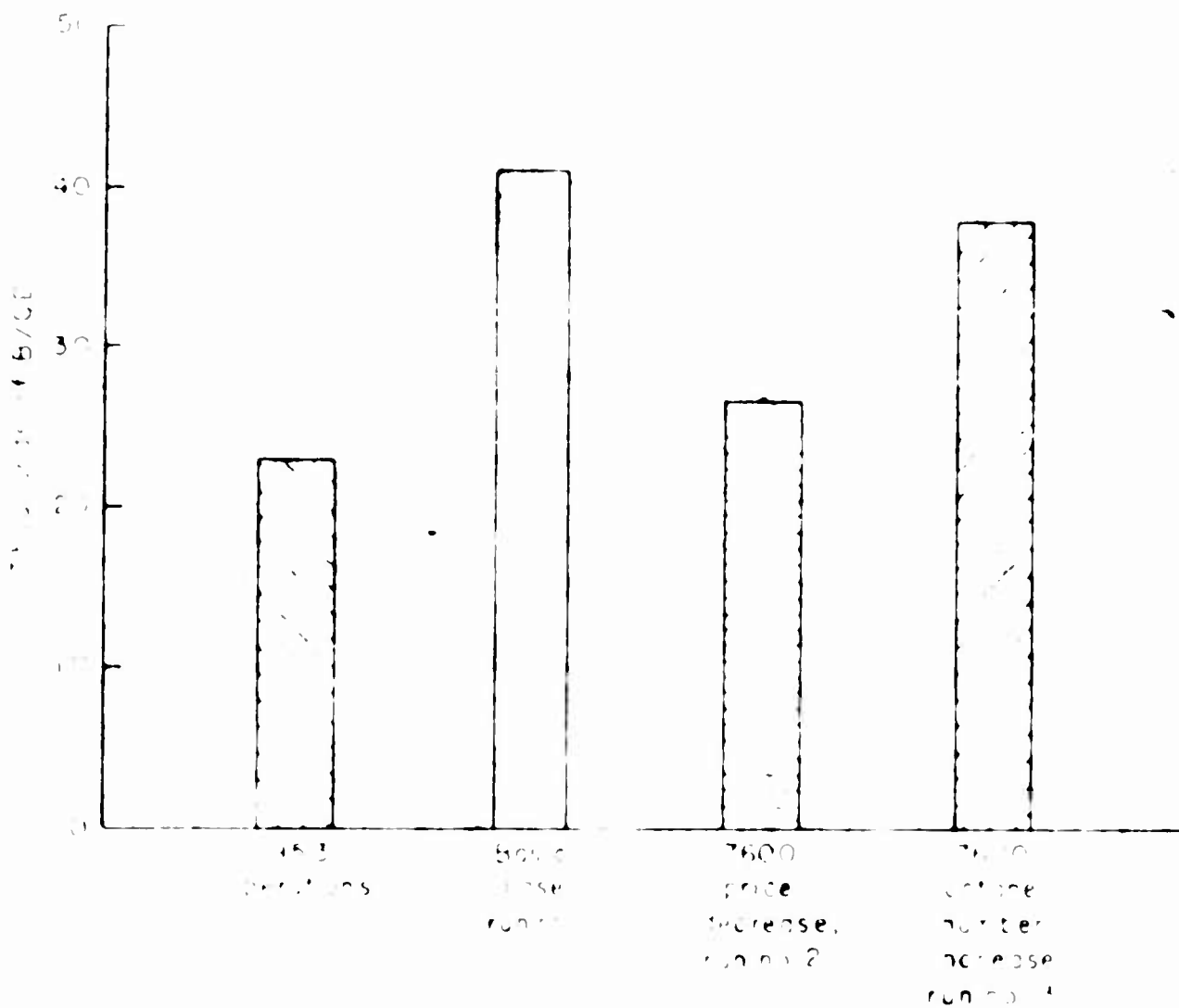| Basic Case Run			Price Decrease Run			Octane No. Increase Run		
600	76	Cutter Stock			Cutter Stock			Cutter Stock
—	1,000	1,1	—	1	1,1	—	1,	1,1
—	1,320		—	2,		—	1,	
1,000	0	—	1,		—	1,		—
2,200	20		1,1	1,		2,	20	
84	380	2,2	1	2		1.	1, 2	
	0	1,3		1,				1,7
110	0				—	11		
120	0	1,1		1,				1,7
600	200		12					1
	400	1						
6,10	0	—		1,1	—	1,		—
3,43	0			1,			1,	
1,	0			2		1,		
3,20	0	—	1,	1,	—	1,		—
1,	0		1,	1,		1,		
11,2	0		,	1,1		11,		
,	0		,			,		
1,		—	1,		—	,		—
	6,1			,			3,1	
41,2	1,670	6, 8	2,	3,	1,	,	1,22	
1,2	2,		1	1,				
	,0							
	10 .00							
100,	2,1	—	,	2,	—			—
	0	—			—	2		—

Fig. 14a — Total payoff, g(x), four cases

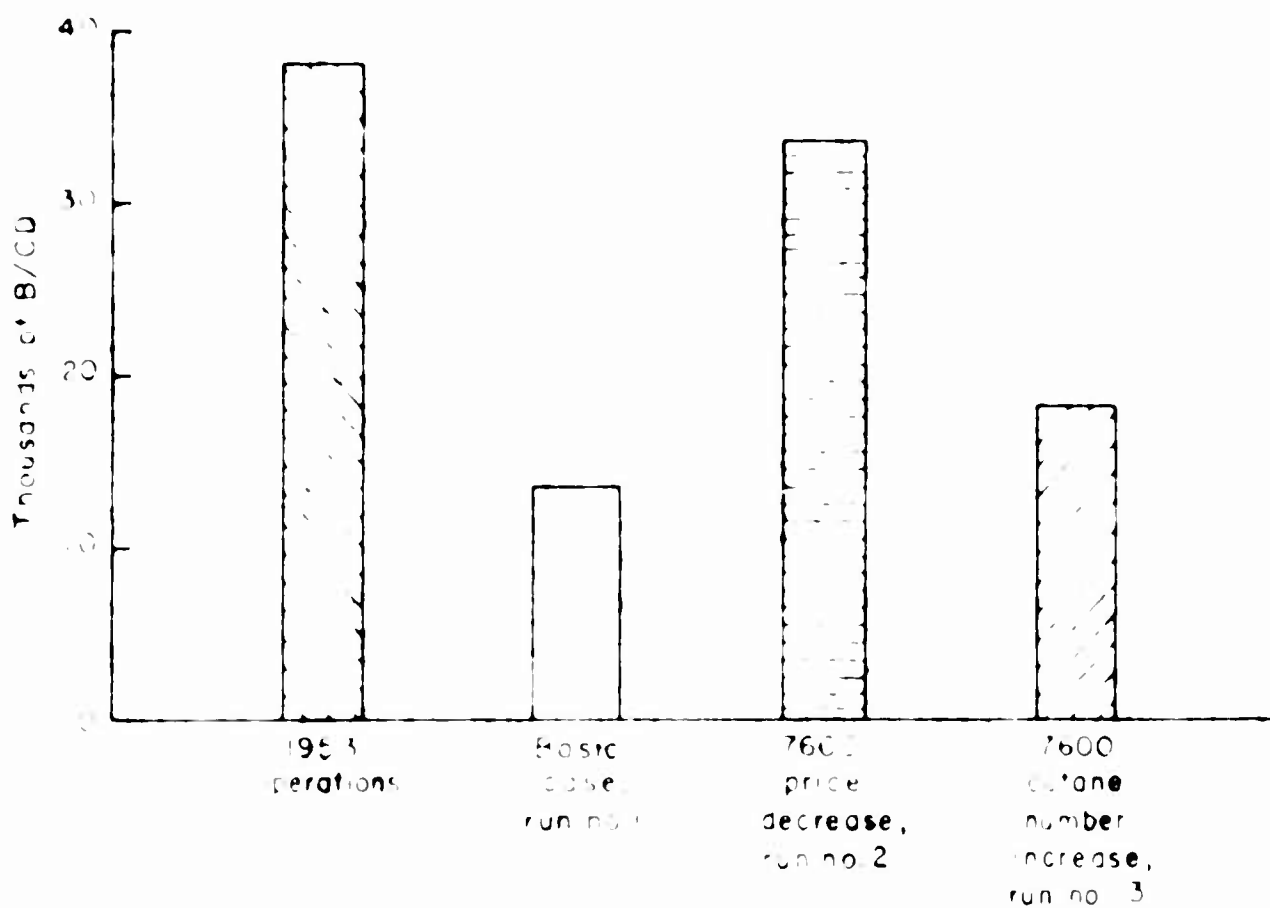Fig. 14b — Total output of 7600 gasoline, four cases (thousands of B/CD)

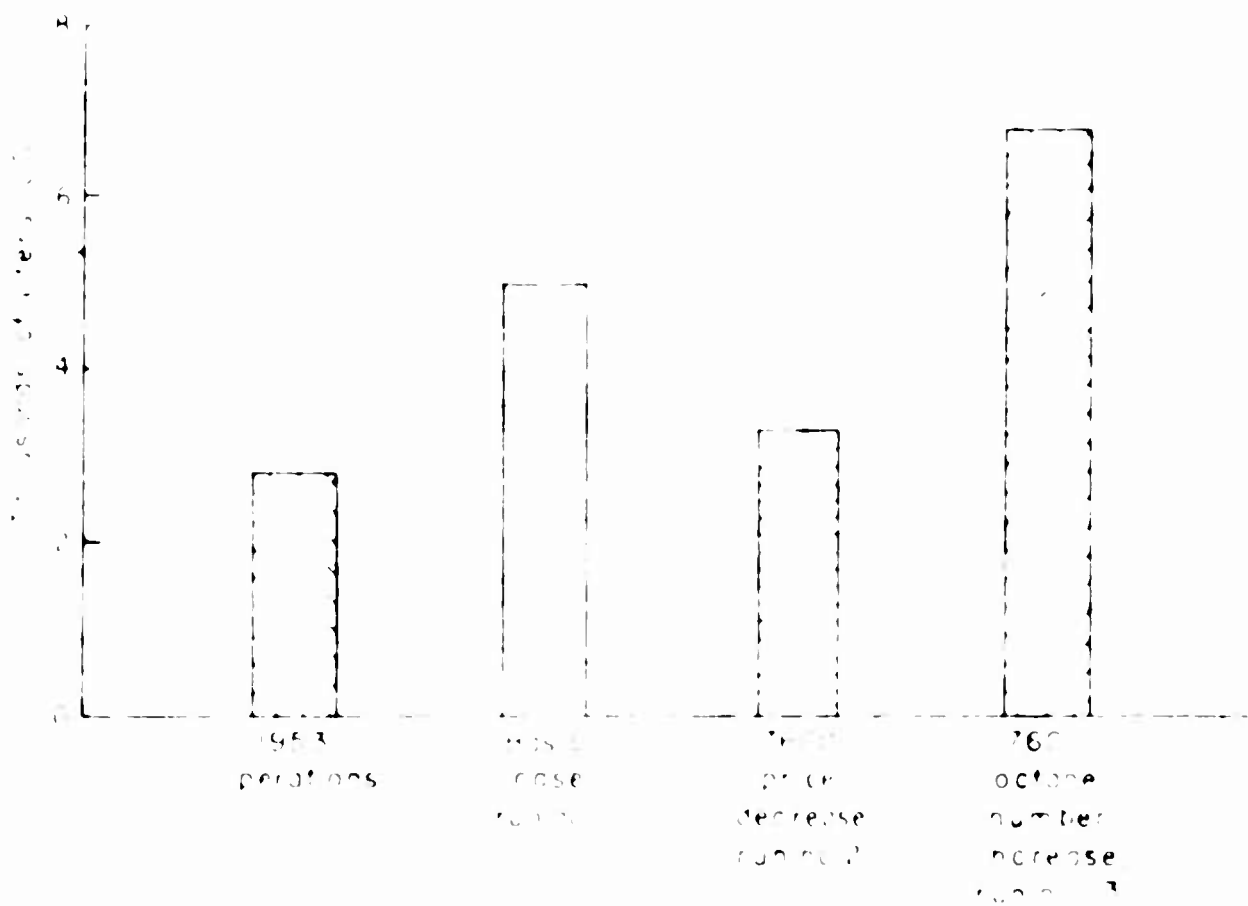Fig 14c — Total output of 76 gasoline, four cases
(thousands of B/CD)

Fig 14 d—Total consumption of TEL, four cases
(thousands of liters/CD)

Unhappily, the life of the operations analyst does not run a
simple course. From the very beginning of the study, it had been
acknowledged that the company could not realistically expect to sell
unlimited quantities of premium gasoline at the prevailing refinery
realization. At the same time, it had not been fully appreciated
how large an increase in the production of this item might be in
order. (The output in the one case is almost double that of the
other.) For this reason, it seems desirable to incorporate an
additional formal restriction into the problem, and to require that
the total production of premium product remain below some upper limit,
L_1. In this way, it would be possible to determine how large an increase in
profits is attainable, even under the limitation of current sales
of the one product. By varying this upper limit parameter,
the analyst could
trace out the effects of alternative sales volumes
upon the total payoff.

Rather than revise the computing setup in order to impose a direct
marketing restriction of this nature, it seemed a more economical
alternative to calculate a second point along the company's "supply
curve" for premium gasoline. That is, for run #2 all parameters were
maintained at the same level as in run #1, except for the unit reali-
zation upon 100 gasoline. This was dropped to 88.42% of its previous
level. The change had the effect of reducing the differential between
premium and regular grade gasoline to less than half of its previous
amount. This alteration in the price relations had a result analogous
to that of a direct quantity limitation upon the make of premium gaso-
line. The optimal output of this product was reduced from 41,250 B/CD

down to 26,385, while the total of regular grade 76 gasoline went up
from 13,670 B/CD to 33,590. In this second case, not only premium
gasoline but also cutter stock production declines. It now becomes
profitable to incorporate most of the heavy gasolines into motor fuel
rather than to blend them into residual fuel oil.

The results of run #2 are equivalent to those that would be
derived by employing an outright sales limitation of 26,385 B/CD. The
unit price imputed to 7600 -- 88.42¢ -- indicates the marginal oppor-
tunity cost of this product. In other words, if the marketing restriction
were raised by one barrel from 26,385 B/CD, and if materials were avail-
able for making this extra barrel, the company would be able to obtain
an additional payoff of 88.42¢ of the current average realization on
7600 gasoline. In the jargon of economists this value is termed the
"shadow price" of the premium product. Evidently, there is a marked
gap here between the shadow price and the average realization upon the
product -- a matter of considerable interest to A. P. Lerner and his
followers. [26] This group has stressed the fact that such discrepancies
are symptomatic of market restraints. At the same time, in them-
selves these discrepancies constitute no proof of either the absence
or the presence of workable competition. [27] From the viewpoint of
internal management, they do provide the company with an indication

[26] See especially A. P. Lerner, "The Concept of Monopoly and the Measure-
ment of Monopoly Power", Review of Economic Studies, June 1934, I,
pp. 157-175.

[27] For a discussion of this concept, see J. M. Clark, "Toward a Concept
of Workable Competition", American Economic Review, June, 1940, Vol. XXX,
No. 2, pp. 241-256.

of the gross gains that might be achieved by reorienting sales policy.

One of the favorite avenues for sales rivalry in the gasoline industry consists of alteration in the specifications for the house-brand product. These modifications occasionally take the form of changes in the volatility index, but more typically center about the octane ratings. For this reason, it was of particular interest to explore the effects of an octane number increase upon the optimal blending schedule and upon the payoff. In run #3,   it was assumed that the specification of the premium gasoline had been increased by 2.5 octane points -- a plausible jump in this requirement. Referring back to the bar charts, Figures 14.a. - 14.d., the reader can compare the results with the basic case: a 2.54% decrease in total payoff, a moderate decrease in the output of premium 7600 fuel, and an increase in the output of 76 gasoline. The make of cutter stock remains the same magnitude in both cases.

The outstanding difference between this situation and the others consists of the high ethyl fluid consumption. It amounts to 6,755 liters/day here -- about a one-third increase as against 5,026 liters/day in the basic case. The TEL concentration level is directly up at the prescribed limit of 3.00 ml./gallon for the premium grade product, but despite this fact, it is only at 2.60 ml. for regular gasoline. Like the results of the reformer analysis in chapter IV, these computations do not yield much comfort to those who would put faith in an often-quoted rule of thumb -- that the optimum blending policy is to assign the maximum allowable TEL concentration to each gallon of product.

Only in making premium grade gasoline in run #3, and in making the
regular product in run #1 was the 3.00 ml. limitation approached.
In all other cases, it paid to employ less than this concentration.

Not only is the ethyl fluid consumption closely connected with
the octane specification of the two motor fuels. It is also sensi-
tive to a variation in the price structure for premium grade gasoline.
Both in the 1953 operations and in run #2, the 7600 product commands
a low shadow price, and by comparison with the basic case, there is
accordingly a low consumption of ethyl fluid. Runs #1 and #3

indicate that even during the ordinary peacetime course of
market conditions, there is a high degree of flexibility possible in
the use of this item. During wartime emergencies, it follows that
there are substantial opportunities for cutting down on the use of
the material in motor fuel in order to release TEL for aviation blends.

An interesting sidelight to the motorist: Under the schedule
set up in the basic case, there is actually a lower TEL concentration
in the 7600 "ethyl" gasoline than in the regular grade product. This
does not mean that the consumer fails to get his money's worth on the
premium product. The ethyl fluid concentration by itself is meaning-
less as an indicator of gasoline quality. In order to determine the
knock rating, it is also necessary to specify the composition of the
product -- the relative amount of components derived from straight-run
gasoline, from catalytic cracking, from thermal reforming, and so on.

5.2. Interpretation of results -- Lagrangean multipliers

Because of the need for withholding cost and revenue figures,
it is not possible to indicate the precise numerical values for the

ten Lagrangean multipliers, u_i. Nevertheless, the general nature of several results may be mentioned. Six of the Lagrangean multipliers turned out to be **positive in all three runs: u_1, the material balance for stock #42;** u_4, the material balance for stock #53; u_5, the octane number of 7600 gasoline; u_6, the octane number of 76 gasoline; u_9, the volatility index of 7600; and u_{10}, the volatility index of 76. **Since the Lagrangean multipliers were always positive in these six cases, the restraint conditions were always satisfied exactly. It never paid to give away octane numbers or volatility index points, and it never paid to employ stock #42 or #53 as cutter.**[28]

A rather different general result held true for u_7 and u_8 -- the "shadow prices" associated with the sulfur content of the two finished gasolines. These two u_i consistently turned out to be zero, for **with the blending stocks currently available, the company as a whole maintained a comfortable position with respect to meeting its sulfur** specifications. This result had not been foreseen in advance, and it turned out to be fortunate that the requirement was phrased in terms of a maximum sulfur content rather than an exact one. Had it been specified as an exact requirement, a distinctly non-optimal program might have been generated.

Concerning the two material balance equations associated with stocks #43 and #52, no general rule can be laid down. Both u_2 and u_3 turned out to have zero values in runs #1 and #3, but achieved

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[28] Table V.3 shows certain quantities of stock #53 used as cutter in the case of runs #2 and #3. This is non-optimal, but is a natural consequence of the approximative type of solution.

positive levels in the case of run #2. That is, for an optimal
solution under the parameters of the second run, neither of these
stocks should be employed as cutter material.

The u_i are of interest, not only in indicating whether or not
the restraint conditions were satisfied as exact equalities, but
also as a method for evaluating the use of additional blending
material in either of the two motor fuel end products. These vari-
ables establish a scale for determining the dollar bonus or the
penalty that should be attached to small incremental quantities of
a material that either exceeds or falls short of the finished pro-
duct specifications. The first use of these Lagrangean multipliers
was to test the earlier assumptions as to which blending stocks could
be excluded from consideration as components for 7600 and which others
could be ruled out as components in the 76 gasoline product. Under
the existing configuration of blending stocks and market conditions,
these assumptions did check out.

It is expected that the Lagrangeans will also be of use during
the interval before the company is in a position to schedule the
blending problem simultaneously with the intra-refinery cracking and
conversion operations. The company's Los Angeles refinery, for
example, has a polyformer unit that has economic features similar
to the thermal reformer discussed in Chapter IV. As with the Kellogg
Company's reformer, there is a continuing problem of selecting the
optimum operating conditions. The higher the reactor temperature,
the better will be the octane rating of the polyformer gasoline,
but the ∧ lower will be the ∧ quantity yielded. The outstanding difference between

the two problems is the fact that the one reformer was considered as

an isolated piece of conversion equipment, and that the other forms

just a single component within an integrated company. In this second

case, because of the possibilities for blending, there is no simple

linkage between the market realization on the end products and the

operations of the one unit. Ideally, the company-wide optimization

                                           doubtful whether this

should be treated as a single calculation, but it is   ∧
will be achieved

∧within the foreseeable future,   Falling

short of this ideal, it would nevertheless be a useful step to attack

the polyformer unit problem through the shadow prices derived from the

blending calculation. Theory assures us that this is a legitimate

assumption if the company, as a whole, is at an optimal point in a

concave programming problem. Theory does not assure us of this result

if the company fails to be in equilibrium, but it strongly suggests

that the same principle holds true even when the company's current
                                          29

position is not an equilibrium one.

5.3 Company policy implications

This much is already evident from the computations: that it
          have paid

would   ∧for Union Oil to forego several millions per annum if a sales
            have been   for disposing
strategy could   ∧devised   ∧of an

increased volume of premium gasoline. For this reason, the problem was

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29. For a discussion of the potentialities for decentralized decision-making through the use of shadow prices, see Oskar Lange, On the Economic Theory of Socialism, Minneapolis, 1938, and also A. P. Lerner, Economics of Control, New York, 1944. Also see C. J. Hitch, "Sub-optimization in Operations Problems", Journal of the Operations Research Society of America, Vol. I, No. 3, May, 1953, pp. 87-99; and A. W. Marshall, "A Mathematical Note on Sub-optimization", ibid., pp. 100-102.

taken up with the company's sales service manager. A number of

alternatives were discussed with him: price reductions, quality

improvements, and an advertising campaign. He agreed that all of

these steps could conceivably increase the sales of premium grade

gasoline, and that the potential increase in revenues might exceed

the extra costs that were entailed. He was particularly concerned

with the difficulties_{of forecasting}

the response of consumers to such changes in strategy,

but did not bring up the parallel question of how the company's

immediate rivals might react.

Rather than risk a major departure in price policy, he indicated

that it would be useful for_{Union Oil}   to explore the possibility

of altering its retailer gasoline discount structure. On any sales

in excess of certain goals, the service station operators might be

allowed an additional discount from the tank wagon price. By read-

justing the product quotas and the sales bonuses, he believed that

it would be feasible for the retail operators to shift sales in the

direction of 7600 and away from 76 gasoline. This policy is one

that need not immediately provoke retaliation from the company's major

rivals, and it does not depend upon a large-scale publicity campaign.

There is the added advantage that the strategy can be altered at

short notice without making consumers aware of a deliberate attempt

to influence their choices.

Even aside from major departures in company-wide sales policy,

the gasoline blending model yielded one result that_{was}   expected to

produce an immediate reduction in cost. In all three computing runs,
it turned out to be optimal to cut motor gasoline component #31 at
the lowest possible end point, i.e., at $300°$ rather than the current $332°F$.
This change in operating practices could be effected without making
any radical alteration in the end-item product mix. At the same
time, the improvement in payoff that would result from this single
step amounts to a sum which, in less than two days, could justify
the entire computing machine expense involved in the study.

6. <u>Summary and conclusions</u>

~~Employing~~ the assumption ~~of~~ fixed availabilities of raw blending materials and of fixed realizations upon finished products, the Union Oil Company's gasoline blending operations were formulated in terms of a constrained-maximum problem.

The outstanding result of the numerical analysis was ~~the fact~~ that it would be worth several millions of dollars per annum if the company were able to shift sales so as to sell additional quantities of premium grade gasoline. As a corollary of this, it follows that the company has considerable latitude in its ability to shift production between the two grades of motor fuel. Marketing problems aside, the ideal pattern of production is quite sensitive to the price structure that prevails.

A second result of the calculations was the finding that the company could increase the octane specification of its premium product, and that even after such a change, the optimal production rate of this one item would remain in excess of current sales.

Neither of these results leads to the implication that the company "ought" to change its price policy or its product specifications. Major changes of this nature cannot be evaluated unless the company's executives also take some account of the response of consumers and of other petroleum refiners. Readily granting the uncertainty that must attend any such conjectures, the formal mathematical treatment seems like a promising approach to the blending problem. It did not turn up with solutions that are absolutely unobtainable by the customary trial-and-error techniques. Rather, it brought to light numerous assumptions

that had previously lain hidden, and holds out the possibility for
closer coordination between the manufacturing operations and the
activities of other branches of the corporation. It is particularly
important for the sales group to have quantitative information avail-
able on the possibilities for shifting between various classes of
refined products, and for changing the specifications of those products.
It is equally significant for them to know the cost implications of
such changes.

Experience within the Union Oil Company suggests that mathematical
models have considerable utility in dealing with complex production
problems. This study also indicates that economists and businessmen
alike are a long distance from having any satisfactory theory of how
a market will respond to a change in the company's sales policies.
In the absence of such knowledge, it is dangerous to place too much
reliance on formal techniques. This same danger exists, though, with
back-of-the-envelope calculations. In none of these matters can a
mathematical model make important business decisions by itself. The
most that can be claimed is that it will summarize the volume of
information that executives have to act upon.